Barysymmetric Multiwavelets on Triangle

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Abstract

In this paper, we give explicit construction of multiwavelets on polygonal region in $\mathbf{R}^2$ that is associated with a nested triangular tessellation. Two different constructions will be presented. The first construction is very similar to Alpert's construction in [3], but unlike the latter 1-D construction, in which case symmetry of basis functions comes in almost automatically, the multiwavelets from our first construction possess no symmetry of any sort. We define a form of symmetry for functions that "live" on triangles, which we call *barysymmetry*, and establish various results about it. We then show that by sacrificing some vanishing moments in the first constructions, we can construct multiwavelets which possess barysymmetry.

Typical members of the bases from both constructions have at least $M > 0$ vanishing moments, but are discontinuous. We discuss how to apply moment-interpolation schemes to improve these orthonormal bases, which gives rise to their smooth biorthogonal counterparts.

1 Introduction

Wavelet is typically designed on simple domains like $\mathbf{R}^n$ or rectangular subset of $\mathbf{R}^n$. Motivated by applications on more general domains, for example, compression of data defined on the sphere or more general surfaces [12], we tackle the problem of designing multiresolution analysis on general manifolds. In this paper, we make some preliminary attempts by constructing multiwavelets on triangulated domains in $\mathbf{R}^2$. Wavelets are invariably designed with the aid of Fourier analysis, which is easily accessible in simple domain like $\mathbf{R}$. On complicated domains Fourier analysis becomes less accessible and we need to explore design methodologies which are less dependent on Fourier means.

Alpert's design of multiwavelets in [3] and their smoother biorthogonal counterparts [10] are only dependent on simple polynomial operations, e.g. Gram-Schmidt orthogonalization and polynomial interpolation, and hence possess high potential to be generalized to more complicated domains. In section 2 we revise the construction of Alpert multiwavelets in 1-D [3] and their smooth biorthogonal counterparts by Donoho, Dyn, Levin and Yu [10]. In Section 3 we describe how to generalize Alpert's construction to triangles. Unlike the 1-D case where symmetry of the resulted basis functions come in automatically, the resulted basis functions from the direct generalization possess no symmetry of any kind. In Section 4, we define a form of symmetry for functions defined on triangles, which we will call *barysymmetry*. We then show that one can construct multiwavelets which possess barysymmetry, with the cost of sacrificing some of the vanishing moments gotten in the first construction. In section 5, we derive the filter banks associated with the multiwavelets constructed in this paper. We end this paper by presenting some of the possible generalizations in Section 6.

2 Alpert Bases in $\mathbf{R}^1$ and their smooth bi-orthogonal duals

Let $\chi^\ell(x)$ denote the l-th Legendre polynomial for the interval $[0,1]$, mutilated to have support $[0,1]$. Thus

$$\begin{aligned} \chi^0(x) &= 1_{[0,1]}(x) && (2.1)\\ \chi^1(x) &= \sqrt{12}(x-1/2)1_{[0,1]}(x) && (2.2)\\ \chi^2(x) &= \sqrt{180}((x-1/2)^2-1/12)1_{[0,1]}(x) && (2.3)\end{aligned}$$

etc. For j and k integers, let $I_{j,k}=[k/2^j,(k+1)/2^j)$ denote a typical dyadic interval, and define

$$\chi^\ell_{j,k} = 2^{j/2}\chi^\ell(2^jx-k), \tag{2.4}$$

the translate and dilate of χ^ℓ that "lives" on $I_{j,k}$ and has $L^2(\mathbf{R})$-norm 1. Any two such functions $\chi^\ell_{j,k}$, $\chi^{l'}_{j,k'}$ with the same scale index j and different $k\neq k'$ or $l\neq l'$ are orthogonal.

Fix $m>0$ and set

$$\overline{V}_j = \overline{V}^{(m)}_j = \text{Span }(\chi^\ell_{j,k} : k\in\mathbf{Z}, 0\le \ell < m)$$

$\overline{V}_j$ is the collection of all piecewise polynomials which are of degree $< m$ on intervals $I_{j,k}, k\in\mathbf{Z}$.

The operator $\overline{P}_jf=\sum_{k\in\mathbf{Z}}\sum_{l=0}^{m-1}\langle f,\chi^\ell_{j,k}\rangle\chi^\ell_{j,k}$ gives orthogonal projection onto $\overline{V}_j$. It is convenient below to adopt the convention that $\chi_{j,k}$ denotes the vector of functions $(\chi^\ell_{j,k})_{\ell=0}^{m-1}$, and that $\langle f,\chi_{j,k}\rangle$ is a vector of inner products. Then we can write $\overline{P}_jf=\sum_k\langle f,\chi_{j,k}\rangle^T\chi_{j,k}$, avoiding the summation over the ℓ index.

The spaces $\overline{V}_j$ are nested: $\overline{V}_j\subset\overline{V}_{j+1}$; indeed a piecewise polynomial with pieces $(I_{j,k})_k$ is also a piecewise polynomial on the finer collection of pieces $(I_{j+1,k})_k$. It follows that the basis elements $(\chi^\ell_{j,k})_{k,l}$ all have a representation in terms of elements $(\chi^\ell_{j+1,k})_{k,l}$ for $\overline{V}_{j+1}$. In vector notation we write this as

$$\chi_{j,k} = H_0\chi_{j+1,2k} + H_1\chi_{j+1,2k+1} \tag{2.5}$$

where the H_i are m by m matrices with entries $(H_i)_{\ell,\ell'}=\langle\chi^\ell_{0,0},\chi^{\ell'}_{1,i}\rangle$. $0\le\ell,\ell'<m$, $i=0,1$.

In the scalar case $m=1$, the H_i are scalars and $H_0=H_1=1/\sqrt{2}$. In the simplest vector case $m=2$

$$H_0 = \begin{bmatrix} \tau & 0 \\ -\delta & \eta \end{bmatrix} \qquad H_1 = \begin{bmatrix} \tau & 0 \\ \delta & \eta \end{bmatrix}$$

where $\tau = 1/\sqrt{2}$, $\delta = \sqrt{3/8}$, $\eta = 1/\sqrt{8}$. (software is available for the general case $m > 2$.

We consider now the detail space $\overline{W}_j \subset \overline{V}_j$ and associated projection $\overline{Q}_j f = \overline{P}_{j+1} f - \overline{P}_j f$. $\overline{W}_j$ consists of piecewise polynomials with pieces $(I_{j+1,k})_k$ which are orthogonal to piecewise polynomials with coarser pieces $(I_{j,k})_k$. An orthogonal basis $(h^\ell_{j,k})_{k,l}$ for $\overline{W}_j$ can be constructed as in Alpert [1]. As $\overline{W}_j \subset \overline{V}_{j+1}$, each $h^\ell_{j,k}$ can be expressed in terms of $(\chi^{\ell'}_{j+1,k'})_{k',\ell'}$:

$$h_{j,k} = G_0 \chi_{j+1,2k} \; + \; G_1 \chi_{j+1,2k+1} \tag{2.6}$$

(in vector notation). In the case $m = 1$, the G_i are scalars: $G_0 = 1/\sqrt{2}, G_1 = -1/\sqrt{2}$. In the case $m = 2$, the G_i obey

$$G_0 = \begin{bmatrix} \eta & \delta \\ 0 & \tau \end{bmatrix} \qquad G_1 = \begin{bmatrix} -\eta & \delta \\ 0 & -\tau \end{bmatrix}.$$

The functions $h^\ell_{j,k}$ are piecewise polynomials supported in $I_{j,k}$ with knots at the endpoints and midpoint of $I_{j,k}$. For pictures see [3] or [2]. We mention that each basis functions from Alpert's construction is either symmetric or antisymmetric.

A key fact about the two-scale matrices we have just defined is that the $2m \times 2m$ matrix

$$U = \begin{bmatrix} H_0 & H_1 \\ G_0 & G_1 \end{bmatrix}$$

is orthogonal. This is equivalent to saying we have two different orthogonal bases for $\overline{V}_1$: $\{(\chi^\ell_{1,k})_{k,\ell}\}$ and $\{(\chi^\ell_{0,k})_{k,\ell}, (h^\ell_{0,k})_{k,\ell}\}$.

Members of Alpert bases are discontinuous, [10] provided an improvement of the Alpert bases via moment-interpolation, which gave rise to their smooth bi-orthogonal counterparts. Moreover, symmetry of the Alpert bases is preserved in this process.

3 Nonseparable construction in $\mathbf{R}^2$

Assume we start from a polygon $R \subset \mathbf{R}^2$ that can be triangulated by $\{T_{0,k} | k \in \mathcal{I}\}$, i.e.

$$R = \cup_{\text{disjoint}} T_{0,k}.$$

R could be a finite or infinite subset of $\mathbf{R}^2$. Consider successive refinement $\{T_{j,k} | j > 0, k \in \mathcal{I}^j\}$ where each triangle $T_{j,k}$ in a finer scale is constructed from one in a coarser level by midpoint subdivision, denoted by

$$T_{j,k} = T_{j+1,k_0} \cup T_{j+1,k_1} \cup T_{j+1,k_2} \cup T_{j+1,k_3}.$$

Note that when $|\mathcal{I}| < \infty$, $|\mathcal{I}^j| = 4^j |\mathcal{I}|$. For consistency, we let $\mathcal{I}^0 = \mathcal{I}$. By convention, we index the center and the three peripheral subtriangles in the mid-point subdivision of $T_{j,k}$ by T_{j+1,k_0}, T_{j+1,k_1}, T_{j+1,k_2} and T_{j+1,k_3} respectively; and when no confusion occurs, simply by T_0, T_1, T_2, T_3. We also denote the edges of T_{j+1,k_0} by $e^1_{j,k}$, $e^2_{j,k}$, $e^3_{j,k}$ or simply by e_1, e_2, e_3, so that e_i represents the common boundary of T_0 and T_i.

Define

$$\begin{aligned} P_d &= \{x^i y^j \mid i+j \le d\} \text{ and} \\ P_d(T) &= \{f \mid f|_T \in P_d \;,\; f|_{\mathbf{R}^2 \setminus T} = 0\} \end{aligned}$$

We have $\dim(P_d) = \dim(P_d(T)) = M = \frac{(d+1)(d+2)}{2}$. We will also denote by $\langle \cdot, \cdot \rangle$ the usual innerproduct in $\mathbf{L}^2(\mathbf{R}^2)$. Define $\langle \cdot, \cdot \rangle_T$ by $\langle f, g \rangle_T = \int \int_T f \; g \; dx \; dy$. (Of course, if either f or g is compactly supported at T, then $\langle f, g \rangle = \langle f, g \rangle_T$.)

Let $T = T_0 \cup T_1 \cup T_2 \cup T_3$ be the midpoint subdivision of an arbitrary triangle T. Denote

$$\overline{V} = P_d(T_0) \oplus P_d(T_1) \oplus P_d(T_2) \oplus P_d(T_3)$$

and

$$\overline{W} = \overline{V} \ominus P_d(T).$$

Note that $\dim(\overline{V}) = 4M$ and $\dim(\overline{W}) = 3M$. Our goal in this section is to develop orthonormal basis for $P_d(T)$ and $\overline{W}$. An orthonormal basis for $P_d(T)$ will generally be denoted by $\{\chi_T^l | 0 \le l < M\}$ or simply by χ_T. The latter notation is understood as a vector of functions. An orthonormal basis of $\overline{W}$ will generally be denoted by $\{h_{e_i}^l | 0 \le l < M \;,\; i = 1, 2, 3\}$ or, simply by the vector notation h_1, h_2 and h_3.

Once such orthonormal basis is constructed, one has orthonormal basis for $\mathbf{L}^2(R)$ of the form

$$\{\chi_{T_{0,k}}, \; h_{e^i_{j,k}} | i = 1, 2, 3, \; j \ge 0, \; k \in \mathcal{I}^j\}, \tag{3.7}$$

such that arbitrary member of $\mathbf{L}^2(\mathbf{R}^2)$ can be decomposed as

$$f = \sum_{k \in \mathcal{I}} \langle f, \chi_{T_{0,k}} \rangle^T \chi_{T_{0,k}} + \sum_{i=1}^{3} \sum_{j \ge 0} \sum_{k \in \mathcal{I}^j} \langle f, h_{e_{j,k}} \rangle^T \; h_{e^i_{j,k}} \tag{3.8}$$

where $\langle f, \chi_{T_{0,k}} \rangle$ is understood as

$$\langle f, \chi_{T_{0,k}} \rangle = [\langle f, \chi^0_{T_{0,k}} \rangle, \cdots, \langle f, \chi^{M-1}_{T_{0,k}} \rangle]^T.$$

and similar interpretation applies to $\langle f, h_{e^i_{j,k}} \rangle$.

Homogeneous expansion (counterparts of Equation 3.8 where the first term of right hand side disappears and the second term has j runs from $-\infty$ to ∞) is also possible, but will enforce impractical uniformity of the triangular tessellation. Therefore we will only consider inhomogeneous expansion in this paper.

MRA. Given a nested triangular tessellation $\{T_j\}$ of $R \subset \mathbf{R}^2$, Let

$$\overline{V_j} = \bigoplus_{k \in \mathcal{I}^j} P_d(T_{j,k}).$$

be the space of *piecewise polynomials* of degree less than d on each of $T_{j,k}$. We have

- $\overline{V}_j \subset \overline{V}_{j+1}$
- $\lim_{j \to \infty} \overline{V}_j = \mathbf{L}^2(\mathbf{R}^2)$
- $\{\chi^l_{T_{j,k}} | 0 \le l < M, k \in \mathcal{T}^j\}$ form an orthonormal basis for $\overline{V}_j$

- Each of $\chi^l_{T_{j,k}}$ is compactly supported at $T_{j,k}$.
- For each $T_{j,k}$, there exists four M by M matrices H_i such that

$$\chi_{T_{j,k}} = \sum_{i=0}^{3} H_i \, \chi_{T_{j+1,k_i}} \tag{3.9}$$

The two-scale relation in Equation 3.9 holds regardless of the choice of $\{\chi^l_{T_{j,k}}\}$, because any function which is piecewise polynomial on $T_{j+1,k}$ is also piecewise polynomial on $T_{j,k}$.

3.1 Barycentric coordinates

Given any (nondegenerate) triangle $T \subset \mathbf{R}^2$ specificed by an *ordered* list of vertices $P_1 = \{x_1, y_1\}, P_2 = \{x_2, y_2\}, P_3 = \{x_3, y_3\}$, one can express any point $P = \{x, y\}$ in $\mathbf{R}^2$ in terms of its *barycentric coordinates* $\tau = (\tau_1, \tau_2, \tau_3)_T$ with respect to T.

$$P = \sum_{i=1}^{3} \tau_i T_i.$$

The τ_i are conventionally made unique by the normalization requirement

$$|\tau| := \sum_{i=1}^{3} \tau_i = 1.$$

If P is inside T, one gets

$$\tau_i \geq 0 \text{ for } i = 1, 2, 3.$$

An important property of barycentric coordinates is their affine invariance: if the triangle T together with P is transformed by an affine transformation Φ, then the barycentric coordinates of $\Phi(P)$ with respect to $\Phi(T)$ is exactly the same as that of P with respect to T. If we think of τ and τ_i as functionals $\tau(P, T)$ and $\tau_i(P, T)$, then the above can be written compactly as $\tau(P, T) = \tau(\Phi(P), \Phi(T))$.

3.1.1 Change of coordinates

Change of coordinates from Cartesian to barycentric coordinates is very easy, one has the transformation rule:

$$\begin{pmatrix} x(\tau_1, \tau_2, \tau_3) \\ y(\tau_1, \tau_2, \tau_3) \end{pmatrix} = \begin{pmatrix} x(\tau_1, \tau_2) \\ y(\tau_1, \tau_2) \end{pmatrix} = \begin{pmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}. \tag{3.10}$$

Hence the Jacobian for the change of coordinates is

$$|\det(\begin{pmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{pmatrix})| = 2|T|.$$

Hence, integral of functions defined on triangles can be calculated easily by

$$\int\int_T f(x, y)\, dx\, dy = 2|T| \int_0^1 \int_0^{1-\tau_1} f(x(\tau_1, \tau_2, 1 - \tau_1 - \tau_2)\ ,\ y(\tau_1, \tau_2, 1 - \tau_1 - \tau_2))\, d\tau_2\, d\tau_1$$

3.1.2 Mutilation of orthonormal basis to arbitrary triangles

Given an orthonormal basis of $P_d(T)$, expressed in terms of barycentric coordinates with respect to T : $\chi^l_T = \chi^l(\tau_1, \tau_2, \tau_3)$, one can immediately write down an orthonormal basis for arbitrary $P(T')$ by:

$$\chi^l_{T'}(\tau_1, \tau_2, \tau_3) = \sqrt{\frac{|T|}{|T'|}}\chi^l(\tau_1, \tau_2, \tau_3) \tag{3.11}$$

where (τ_1, τ_2, τ_3) is interpreted with respect to T' on the left hand side and with respect to T on the right hand side. It may sound confusing at the first glance, but the true power of barycentric coordinates is that it allows us to think of $\chi^l(\tau_1, \tau_2, \tau_3)$ as a "symbol" in terms of the symbol triplet (τ_1, τ_2, τ_3), rather than as a function. For example, when we say "$\chi^l(\tau_1, \tau_2, \tau_3) = \cos(\tau_1) - \tau_2/\tau_3$" without specifying the reference triangle for (τ_1, τ_2, τ_3), then all we can say is that $\chi^l(\tau_1, \tau_2, \tau_3)$ is the *symbol* "$\cos(\tau_1) - \tau_2/\tau_3$", rather than a well defined function $\chi^l : \mathbf{R}^2 \to \mathbf{R}$. The potential ambiguity turns out to be a useful property, because when a user request for an orthonormal basis of an arbitrary $P(T')$, we can then interpret each triplet of symbols (τ_1, τ_2, τ_3) as a physical point in $\mathbf{R}^2$ by letting T' be the reference triangle, and then the symbols $\chi^l(\tau_1, \tau_2, \tau_3)$ become well-defined functions. Followed by proper scalings, they become to form an orthonormal basis for $P_d(T')$.

One can think of "translation" and "dilation" in the 1-D setting be naturally captured by the affine invariance of barycentric coordinates in the 2-D triangular setting. In fact, one can think of Equation 3.11 as just a 2-D version of Equation 2.4.

As a convention (to avoid future confusion, especially in Section 3.2.2 and 3.2.2), when we subscript a function variable by a triangle variable (which in turn must be specified as an ordered list of three points in $\mathbf{R}^2$), e.g. "$f_T(\tau_1, \tau_2, \tau_3)$", it is understood that (τ_1, τ_2, τ_3) are barycentric coordinates relative to triangle T.

3.1.3 Change of reference triangle

If a point P has barycentric coordinates $\tau = (\tau_1, \tau_2, \tau_3)$ relative to triangle T (defined by $\{(x_i, y_i)|i = 1, 2, 3\}$. Then the barycentric coordinates τ' of P relative to triangle T' (defined by $\{(x_i, y_i)|i = 1, 2, 3\}$) can be calculated by:

$$(\tau_1', \tau_2', \tau_3') = (\tau_1, \tau_2, \tau_3)M_{T\to T'}$$

where

$$M'_{T\to T'} = \begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \tag{3.12}$$

3.1.4 Midpoint subdivision

Midpoint subdivision can also be easily described using barycentric coordinates. Let $T = T_0 \cup T_1 \cup T_2 \cup T_3$ be the midpoint subdivision of an arbitrary triangle T. Let (τ_1, τ_2, τ_3) be defined with respect to T, then

$$\begin{aligned} T_0 &= \{(\tau_1, \tau_2, \tau_3)|0 \le \tau_1, \tau_2, \tau_3 \le 1/2\} \\ T_1 &= \{(\tau_1, \tau_2, \tau_3)|\tau_1 \ge 1/2, \tau_2, \tau_3 \ge 0\} \\ T_2 &= \{(\tau_1, \tau_2, \tau_3)|\tau_2 \ge 1/2, \tau_1, \tau_3 \ge 0\} \\ T_3 &= \{(\tau_1, \tau_2, \tau_3)|\tau_3 \ge 1/2, \tau_1, \tau_2 \ge 0\} \end{aligned}$$

Later we will study symmetry of functions defined on triangles, and the relative ordering of vertices in the specification of T_i become relevant. We adopt the convention that if T is specified by the ordered list $T = \{P_1, P_2, P_3\}$, then T_i will be specified by the following ordered lists:

$$\begin{aligned} T_0 &= \{\frac{P_1+P_2}{2}, \frac{P_2+P_3}{2}, \frac{P_3+P_1}{2}\},\ T_1 = \{P_1, \frac{P_1+P_2}{2}, \frac{P_3+P_1}{2}\}, \\ T_2 &= \{P_2, \frac{P_2+P_3}{2}, \frac{P_1+P_2}{2}\},\ T_3 = \{P_3, \frac{P_3+P_1}{2}, \frac{P_2+P_3}{2}\}. \end{aligned} \tag{3.13}$$

3.2 Construction I

We now present a construction of multiwavelets for our setting, which is similar to the construction by Alpert in 1-D.

3.2.1 Scaling Functions

We first seek for scaling functions $\chi^l_{T_{0,k}}$ which are orthogonal polynomials on $T_{0,k}$ and vanish outside so that

$$\langle \chi^l_{T_{0,k}} \chi^{l'}_{T_{0,k}} \rangle_{\mathbf{R}^2} = \langle \chi^l_{T_{0,k}} \chi^{l'}_{T_{0,k}} \rangle_{T_{0,k}} = \delta_{ll'}.$$

Virtually any choice of orthogonal polynomial systems (with respect to the $\langle \cdot, \cdot \rangle_{T_{0,k}}$ inner-product) is valid. Due to our experience in 1-D, we will consider Legendre-typed polynomial basis. Legendre polynomials on triangles can easily be constructed as follows. First of all, due to the comment at the end of last section, we can restrict our attention to the *base triangle* $T = T_B$ with vertices $(0,0), (0,1), (1,0)$. Consider the power basis $\{1, x, y, x^2, xy, y^2, \ldots\}$ which is equal to $\{1, \tau_2, 1-\tau_1-\tau_2, \tau_2^2, (1-\tau_1-\tau_2)\tau_2, (1-\tau_1-\tau_2)^2, \ldots\}$ in barycentric coordinates with respect to T_B. Applying the Gram-Schimitt process to the latter sequence (in the listed ordering) with respect to the inner product $\langle \cdot, \cdot \rangle_{T_B}$ will give us Legendre polynomials $\{\pi^l | l \geq 0\}$ on T_B.

Define, for $l \geq 0$, $\chi^l_{T_B} = \pi^l 1_{T_B}$. The resulted sequence $S = \{\chi^l_{T_B}\}$ becomes a *triangular sequence of orthogonal polynomials* in $\mathbf{R}^2$: for any $d \geq 0$, the first $M = (d+1)(d+2)/2$ elements of S form an orthonormal basis for both $P_d(T_B)$ and $P_d(\mathbf{R}^2)$. Using the procedure described above, we obtain the first few members of S as

$$\begin{aligned} \chi^0_{T_B}(\tau_1, \tau_2) &= \sqrt{2}\ 1_{T_B}(\tau_1, \tau_2) \\ \chi^1_{T_B}(\tau_1, \tau_2) &= (-2 + 6\tau_2)\ 1_{T_B}(\tau_1, \tau_2) \\ \chi^2_{T_B}(\tau_1, \tau_2) &= 2\sqrt{3}(1 - 2\tau_1 - \tau_2)\ 1_{T_B}(\tau_1, \tau_2) \\ \chi^3_{T_B}(\tau_1, \tau_2) &= \sqrt{6}(1 - 8\tau_2 + 10\tau_2^2)\ 1_{T_B}(\tau_1, \tau_2) \\ \chi^4_{T_B}(\tau_1, \tau_2) &= 3\sqrt{2}(-1 + 2\tau_1 + 6\tau_2 - 10\tau_1\tau_2 - 5\tau_2^2)\ 1_{T_B}(\tau_1, \tau_2) \\ \chi^5_{T_B}(\tau_1, \tau_2) &= \sqrt{30}(1 - 6\tau_1 + 6\tau_1^2 - 2\tau_2 + 6\tau_1\tau_2 + \tau_2^2)\ 1_{T_B}(\tau_1, \tau_2) \end{aligned}$$

etc.. [1] See Figure 1 for a graphical display. By Equation 3.11, we can choose the scaling functions as

$$\chi^l_{T_{j,k}} = \sqrt{\frac{1}{2|T_{j,k}|}} \chi^l_{T_B}\ ,\ 0 \leq l < M. \tag{3.14}$$

[1] When a function $f(\tau_1, \tau_2, \tau_3)$ is defined in terms of barycentric coordinates, we abuse the notation by writing $f(\tau_1, \tau_2) = f(\tau_1, \tau_2, 1 - \tau_1 - \tau_2)$.

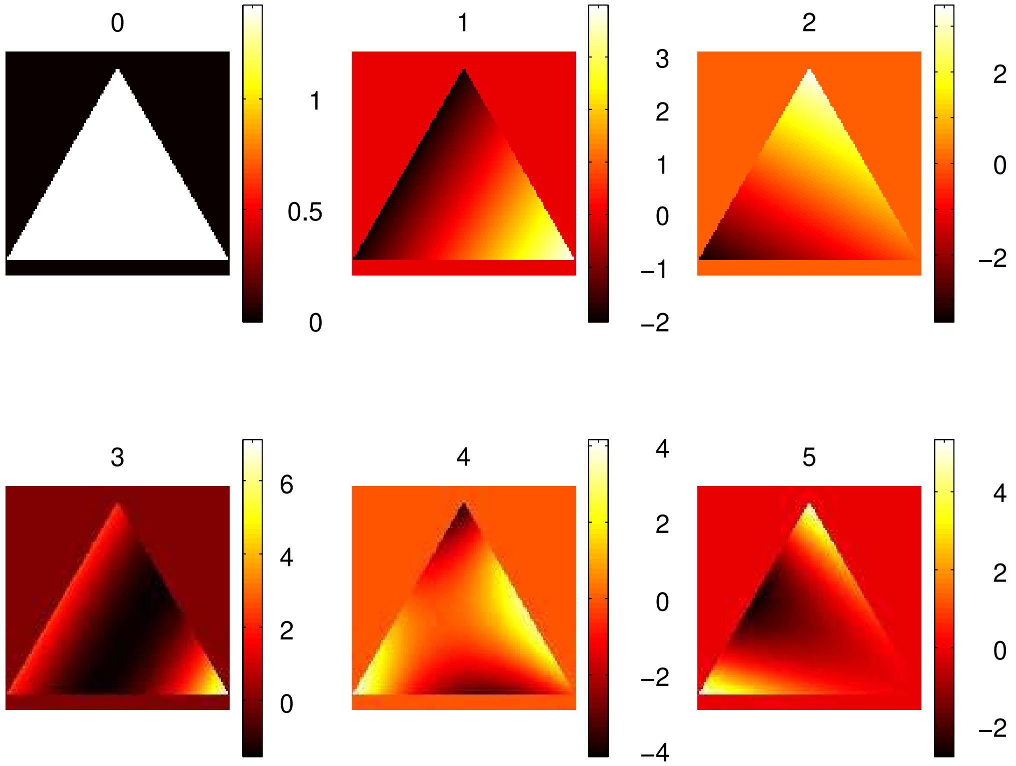


Figure 1: Legendre Polynomials on a triangle, $d \leq 2$

3.2.2 Mother Multiwavelets

Having defined an orthonormal basis for $P_d(T)$, we now present a construction for an orthonormal basis for

$$\overline{W} = (P_d(T_0) \oplus P_d(T_1) \oplus P_d(T_2) \oplus P_d(T_3)) \ominus P_d(T).$$

Before we proceed, the authors would like to mention that the following construction is strongly motivated by the 1-D construction in [3].

Recall that $dim(\overline{W}) = 3M$. We will generally denote by

$$\{h_i^l | i = 1, 2, 3, 0 \leq l < M\}$$

any set of $3M$ functions that can be combined with a basis of $P(d)$ to span $P_d(T_0) \oplus P_d(T_1) \oplus P_d(T_2) \oplus P_d(T_3)$.

Let $\{\pi_T^l\}$ be the legendre polynomials mutilated to the triangle T (c.f. Equation 3.11.) Consider

$$h_i^l(\tau_1, \tau_2, \tau_3) = \begin{cases} \pi_{T_i}^l((\tau_1, \tau_2, \tau_3) M_{T \to T_i}) & \text{on } T_i, \\ -\pi_{T_i}^l((\tau_1, \tau_2, \tau_3) M_{T \to T_i}) & \text{on } T \backslash T_i \\ 0 & \text{on } \mathbf{R}^2 \backslash T \end{cases} \tag{3.15}$$

$i = 1, 2, 3,\ \ l = 0 \ldots (M-1)$ (Note: (τ_1, τ_2, τ_3) in Equation 3.15 is defined relative to the triangle T.) Notice that $\{h_i^l\} \cup \{\chi_T^l\}$ spans $P_d(T_0) \oplus P_d(T_1) \oplus P_d(T_2) \oplus P_d(T_3)$.

We will modify $\{h_i^l\}$ successively to make an orthonormal basis for $\overline{W}$.

We first orthogonalize $\{h_i^l\}$ against $P_d(T)$ by

$$h_i^l \longleftarrow h_i^l - \sum_{l=0}^{M-1} \langle h_i^l, \chi_T^l \rangle \chi_T^l,$$

then $span(\{h_i^l\}) = P_d(T)^\perp \cap \bigoplus_{i=0}^3 P_d(T_i)$. Note also that each member of $\{h_i^l\}$ now has $M = (d+1)(d+2)/2$ vanishing moments:

$$\langle h_i^l, x^p y^q \rangle = 0 \text{ for } p+q \leq d, p, q \geq 0.$$

In principle, all we have to do next is to orthogonalize $span(\{h_i^l\})$ by Gram-Schmidt to get an orthonormal basis for $\overline{W}$. It is, however, possible to arm $\{h_i^l\}$ with more vanishing moments before orthogonalization among themselves.

To do so, we order $\{h_i^l\}$ as

$$\mathcal{L} = [h_1^0, h_2^0, h_3^0, h_1^1, h_2^1, h_3^1, \ldots, h_1^{(M-1)}, h_2^{(M-1)}, h_3^{(M-1)}]. \tag{3.16}$$

In the following, we will generally denote the members of $\mathcal{L}$ by $\mathcal{L} = [h_1, h_2, \ldots]$.

Recall that every member in $\mathcal{L}$ is orthogonal to χ_T^l for $l < M$. Initialize $t = M$, $\chi^{\text{target}} = \chi_T^t$ and $\mathcal{L}' = [\,]$ (empty list.) We apply to $\mathcal{L}$ the following operations:

1. Find the first element, h, in the list $\mathcal{L}$ which is non-orthogonal to χ^{target}, or if no such h exists, repeat: set $t \longleftarrow t+1$ and $\chi^{\text{target}} = \chi_T^t$ until we find such a h.

2. reorder $\mathcal{L}$ as
$$\mathcal{L} \longleftarrow [h, [\mathcal{L} \backslash h]]$$

3. Orthogonalize all but the first function in $\mathcal{L}$ against χ^{target} by
$$h_i \longleftarrow h_i - \frac{\langle h_i, \chi^{\text{target}} \rangle}{\langle h_1, \chi^{\text{target}} \rangle} h_1.$$

4. $\mathcal{L} \longleftarrow [h_2, h_3, \ldots]$, $\mathcal{L}' \longleftarrow [\mathcal{L}', h_1]$. If $\mathcal{L}$ is now an empty list, $t \longleftarrow t+1$ goto 5, else goto 1.

5. $\mathcal{L} \longleftarrow \mathcal{L}'$

Note that $span(\mathcal{L})$ is unaffected by the above operations, but that each h_j now has at least $M + j - 1$ vanishing moments.

Finally we orthogonalize $\mathcal{L}$ among themselves by the Gram-Schmidt process. In order to retain the vanishing moments acquired in the last steps, we must Gram-Schmidt $\mathcal{L}$ in the reverse order.

Again, by Equation 3.11, we can define

$$h_{e_{j,k}^i}^l = \sqrt{\frac{1}{2|T_{j,k}|}} h_i^l \ , \ 0 \leq l < M,$$

assuming $\{h_i^l\}$ is derived on the triangle $T = T_B$. We have completed our first construction for multiwavelets on $\mathbf{L}^2(R)$.

Figure 2 and 3 display the multiwavelets from construction I for $d = 0$ and $d = 1$ respectivelty.

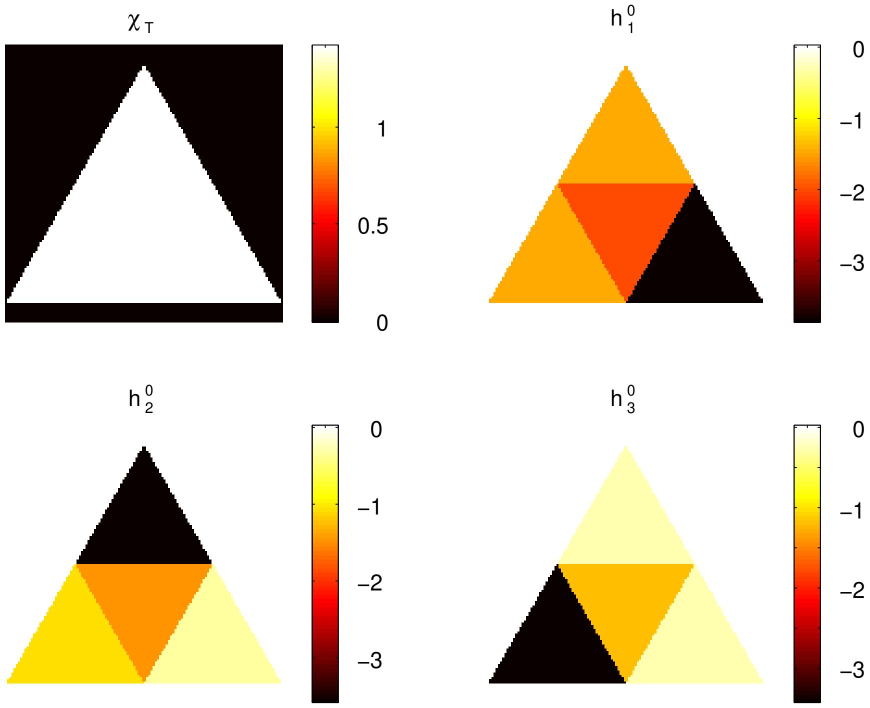


Figure 2: Multiwavelets on triangle from construction I, $d = 0$

4 Restoring Symmetry

While the first construction is a natural generation of the 1-D construction in [3], the resulted multiwavelets from construction I lack a nice property that come in effortlessly in the 1-D case, namely, *symmetry*. As mentioned in Section 2, every member of the Alpert basis is either symmetric or anti-symmetric about the midpoint of its supports. Symmetry in wavelets is desirable in applications such as image coding [4] and curve and surfaces editing [9]. In 1-D, it is well known that compactly supported orthonormal dyadic wavelets based on translation and dilation of a single function cannot possess symmetry, with the Haar basis as a (trivial) exception. See, for example, Chapter 8 of [8] for a proof of this fact. Various solutions have been proposed by wavelet designers to cure this problem, e.g. by trading off between vanishing moments with the amount of symmetry [8] in order to acquire "more symmetry" in wavelets, by bi-orthogonal wavelets[7], by triadic dilation [6], by complex-valued wavelets [13], and, of course, by multiwavelets.

The multiwavelets considered in Section 3.2.2 was constructed with the goal of optimizing the number of vanishing moments, and symmetry was completely disregarded. In the next section we will discuss a form of symmetry on triangles, and show in subsequent sections that multiwavelets can be constructed that possess such kind of symmetry, with the cost of sacrificing some of the vanishing moments acquired in construction I.

4.1 BarySymmetry

For functions that "live" on a triangles, it is less clear what symmetry means. Alternatively, it is easier to think of a set of functions being *mutually symmetric* to each other in some sense, and aim at designing multiwavelet systems with that property. Apparently it is a

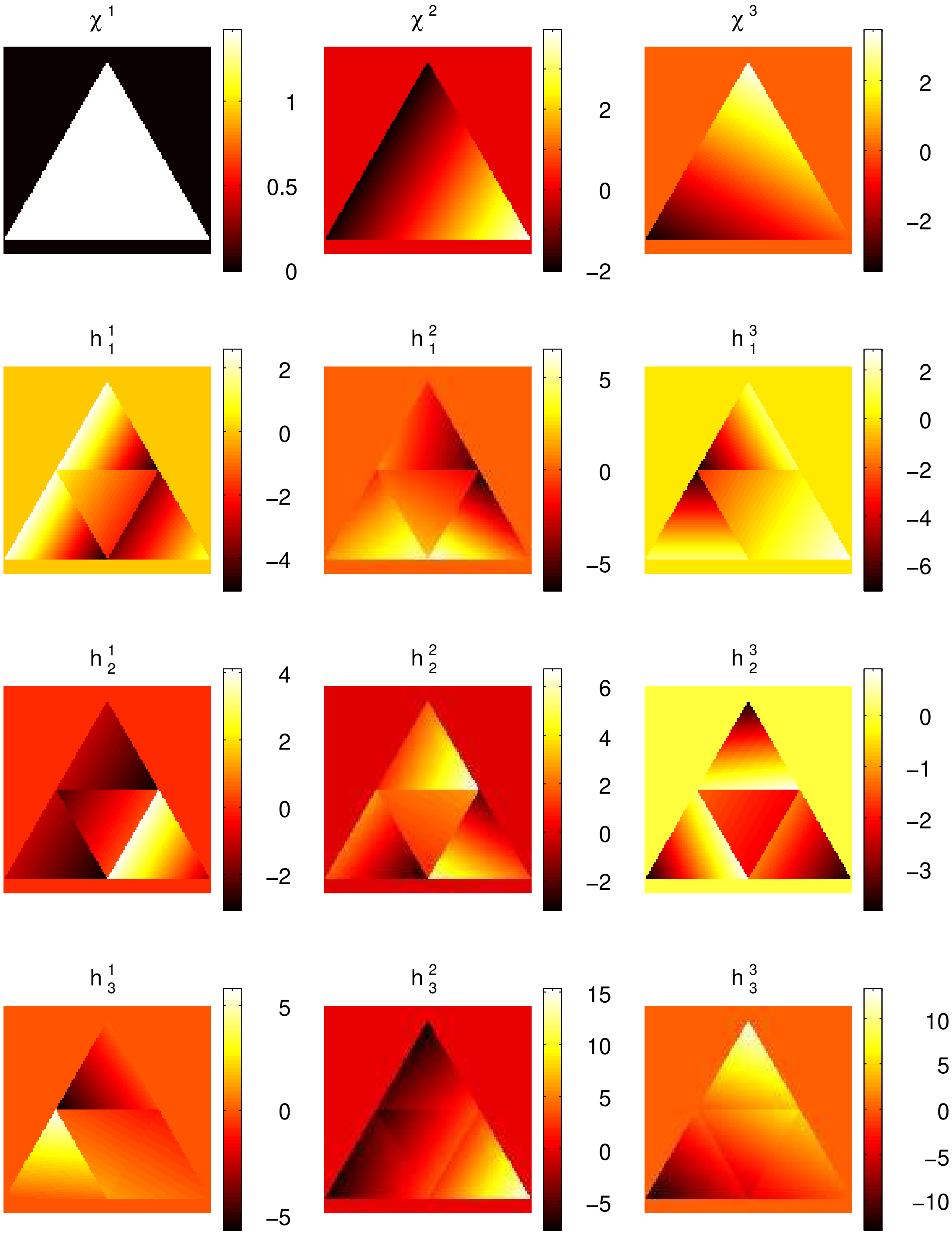


Figure 3: Multiwavelets on triangle from construction I, $d = 1$

weak form of symmetry we are considering. Multiwavelets with this kind of symmetry have been proposed in 1-D by Geronimo-Hardin-Massopust [11], in which a basis element may not be symmetric or antisymmetric, but that a "mirror" of the element is always presented in the basis. Precisely,

Definition 4.1 *A set of function $\mathcal{S} = \{p_i(\tau_1, \tau_2, \tau_3) \mid i = 1 \dots n\}$ is barysymmetric with respect to T if $p_i(\tau_{\sigma(1)}, \tau_{\sigma(2)}, \tau_{\sigma(3)}) \in \mathcal{S}$ for all $\sigma \in CyclicGroup(\{1,2,3\})$ and $1 \leq i \leq n$. (Note: $CyclicGroup(\{1,2,3\}) := \{(1,2,3),(2,3,1),(3,1,2)\}$.)*

We could potentially define a stronger type of barysymmetry if the cyclic group $CyclicGroup(\{1,2,3\})$ is replaced by the full permutation group

$$PermuGroup(\{1,2,3\}) = \{(1,2,3),(2,3,1),(3,1,2),(1,3,2),(3,2,1),(2,1,3)\}$$

in the above definition. We will simply use the term "strong-barysymmetry" for this modified definition.

Definition 4.2 *Given a function $p : \mathbf{R}^2 \to \mathbf{R}$ defined in terms of barycentric coordinates with respect to T, the set $\mathcal{S} = \{p(\tau_{\sigma(1)}, \tau_{\sigma(2)}, \tau_{\sigma(3)}) \mid \sigma \in CyclicGroup(\{1,2,3\})$ is called the barysymmetric orbit of p with respect to T, denoted generally by $Orbit(p)$.*

Lemma 4.3 *For any p, $|Orbit(p)| = 1$ or 3.*

Proof. Assume $|Orbit(p)| = 2$. Without lost of generality, we can assume (i) $p(\tau_1, \tau_2, \tau_3) = p(\tau_2, \tau_3, \tau_1)$ for all (τ_1, τ_2, τ_3) and (ii) there exist a point $(\overline{\tau_1}, \overline{\tau_2}, \overline{\tau_3})$ such that $p(\overline{\tau_2}, \overline{\tau_3}, \overline{\tau_1}) \neq p(\overline{\tau_3}, \overline{\tau_1}, \overline{\tau_2})$. But these immediately lead to a contradiction because (i) implies $p(\overline{\tau_2}, \overline{\tau_3}, \overline{\tau_1}) = p(\overline{\tau_3}, \overline{\tau_1}, \overline{\tau_2})$. Therefore, $|Orbit(p)| = 1$ or 3. □

One can define p be *symmetric with respect to T* if $|orbit(p)| = 1$, but we will focus on barysymmetry in the following construction.

4.2 Symmetric Orthogonalization

In this section, we will prove that under some mild conditions, an arbitrary set of independent and barysymmetric functions $\mathcal{S}'$ can always be orthogonalized into another barysymmetric set $\mathcal{S}'$. The proof of this result is a constructive one, and we will use the ideas in this section extensively in our next construction of multiwavelets.

Given $p \in \mathbf{L}^2(T)$ with $||p||_T := \sqrt{\langle p, p \rangle_T} = 1$, we define

$$\mathcal{P}(p) := \langle p(\tau_1, \tau_2, \tau_3), p(\tau_2, \tau_3, \tau_1) \rangle_T. \tag{4.17}$$

The normalization condition $||p||_T = 1$ implies $-1 \leq \mathcal{P}(p) \leq 1$ (Cauchy-Schwarz inequality.) Moreover, a simple change of variable argument shows that

$$\langle p(\tau_1, \tau_2, \tau_3), p(\tau_2, \tau_3, \tau_1) \rangle_T = \langle p(\tau_2, \tau_3, \tau_1), p(\tau_3, \tau_1, \tau_2) \rangle_T = \langle p(\tau_3, \tau_1, \tau_2), p(\tau_1, \tau_2, \tau_3) \rangle_T. \tag{4.18}$$

In other words, the number $\mathcal{P}(p)$ completely specifies the Gram matrix of $orbit(p)$.

In this paper, we are mainly interested in functions such that

$$|orbit(p)| = \dim(span(orbit(p))). \tag{4.19}$$

Notice that not every function defined on a given triangle satisfies Equation 4.19. Consider

$$p(\tau_1, \tau_2, \tau_3) = \begin{cases} 1 & \text{if } \tau_1 > 1/2, \\ -1 & \text{if } \tau_2 > 1/2, \\ 0 & \text{if } \tau_3 > 1/2, \\ 0 & \text{if } \tau_1, \tau_2, \tau_3 \leq 1/2. \end{cases} \tag{4.20}$$

Then $|orbit(p)| = 3$, but $p(\tau_1, \tau_2, \tau_3) + p(\tau_2, \tau_3, \tau_1) + p(\tau_3, \tau_1, \tau_2) \equiv 0$, which implies $\dim(span(orbit(p))) < 3$. (In fact, $\dim(span(orbit(p))) = 2$.)

Lemma 4.4 *For any $p \in \mathbf{L}^2(T)$ with $||p||_T = 1$ and $|orbit(p)| = \dim(span(orbit(p)))$, we have*

$$-1/2 < \mathcal{P}(p) \leq 1. \tag{4.21}$$

Moreover, $|Orbit(p)| = 1$ if and only if $\mathcal{P}(p) = 1$ (Equivalently, $|Orbit(p)| = 3$ if and only if $-1/2 < \mathcal{P}(p) < 1$.)

Proof.

If $|Orbit(p)| = 1$, then clearly $\mathcal{P}(p) = 1$.

It is well known that if a set of vectors $\{v_1, \ldots, v_n\}$ in an inner-product space is linearly independent, then the Gram matrix $G = [G_{ij}]$, $G_{i,j} = \langle v_i, v_j \rangle$ is symmetric positive definite. Therefore if $|Orbit(p)| = \dim(span(orbit(p))) = 3$, then the Gram matrix of $orbit(p)$ is

$$G = \begin{pmatrix} 1 & \mathcal{P}(p) & \mathcal{P}(p) \\ \mathcal{P}(p) & 1 & \mathcal{P}(p) \\ \mathcal{P}(p) & \mathcal{P}(p) & 1 \end{pmatrix}$$

and

$$\det(G) = (1 + 2\mathcal{P}(p))(1 - \mathcal{P}(p))^2 > 0$$

which happens when and only when $-1/2 < \mathcal{P}(p) < 1$.

To conclude, $|Orbit(p)| = 3$ if and only if $-1/2 < \mathcal{P}(p) < 1$ and $|Orbit(p)| = 1$ if and only if $\mathcal{P}(p) = 1$. □

For the function defined by Equation 4.20, $\mathcal{P}(p) = -1/2$. Therefore, the dimensionality condition Equation 4.19 is necessary.

Lemma 4.5 *Given $p \in \mathbf{L}^2(T)$ that satisfies Equation 4.19, there exists $q \in \mathbf{L}^2(T)$ such that $orbit(q)$ is an orthonormal set with respect to $\langle \cdot, \cdot \rangle_T$ and $span(orbit(p)) = span(orbit(q))$.*

Proof. We can assume without loss of generality that $||p||_T = 1$ (if not, then replace p by $p/||p||_T$.) If $|orbit(p)| = 1$, then we just set $q = p$. Otherwise, $|orbit(p)| = 3$. In order to orthogonalize $|orbit(p)|$ in a barysymmetric fashion, define q by

$$q(\tau_1, \tau_2, \tau_3) = r_1 q(\tau_1, \tau_2, \tau_3) + r_2 q(\tau_2, \tau_3, \tau_1) + r_3 q(\tau_3, \tau_1, \tau_2).$$

So

$$orbit(q) = \{q(\tau_1, \tau_2, \tau_3), q(\tau_2, \tau_3, \tau_1), q(\tau_3, \tau_1, \tau_2)\}.$$

It remains to show the existence of $r_1, r_2, r_3 \in \mathbf{R}$ such that $orbit(q)$ is orthonormal. By Equation 4.18, orthonormality implies only 2 constraints on $(r1, r2, r3)$:

$$||q||_T = 1 \text{ and } \mathcal{P}(q) = 0. \tag{4.22}$$

Denote $\mathcal{P} = \mathcal{P}(p)$, Equation 4.22 can easily be turned into explicit algebraic form:

$$\begin{aligned} 1 &= ||q||_T = r_1^2 + r_2^2 + r_3^2 + 2(r_1 r_2 + r_2 r_3 + r_3 r_1)\mathcal{P} \\ 0 &= \mathcal{P}(q) = (r_1^2 + r_2^2 + r_3^2)\mathcal{P} + (r_1 r_2 + r_2 r_3 + r_3 r_1)(\mathcal{P} + 1). \end{aligned}$$

One set of solution in $\mathbf{C}$ is

$$r_1 = (\sqrt{\frac{1}{1+2\mathcal{P}}} - r_3 + \sqrt{\mathcal{D}})/2, \; r_2 = (\sqrt{\frac{1}{1+2\mathcal{P}}} - r_3 - \sqrt{\mathcal{D}})/2$$

where

$$\mathcal{D} = \frac{2 + 3\mathcal{P} - 3r_3^2 - 3\mathcal{P}r_3^2 + 6\mathcal{P}^2 r_3^2}{(1-\mathcal{P})(1+2\mathcal{P})} + 2\sqrt{\frac{1}{1+2\mathcal{P}}} r_3.$$

Note that $\sqrt{\frac{1}{1+2\mathcal{P}}}$ is well-defined because $\mathcal{P} > -1/2$. It remains to show that for every $\mathcal{P} > -1/2$, we can always find r_3 such that $\mathcal{D} \geq 0$. A careful calculation shows that $\mathcal{D} > 0$ if and only if

$$\sqrt{\frac{1}{1+2\mathcal{P}}} - \frac{2}{3}\frac{\sqrt{(1-\mathcal{P})(1+2\mathcal{P})^2}}{(1-\mathcal{P})(1+2\mathcal{P})} < r_3 < \sqrt{\frac{1}{1+2\mathcal{P}}} + \frac{2}{3}\frac{\sqrt{(1-\mathcal{P})(1+2\mathcal{P})^2}}{(1-\mathcal{P})(1+2\mathcal{P})} \tag{4.23}$$

and the above range is non-empty since $-1/2 < \mathcal{P} < 1$. (In fact, one can verify that $r_3 = 1$ is always a valid choice as long as $-0.47 < \mathcal{P} < 1$.) □

Lemma 4.6 *Given P, Q be two barysymmetric sets in $\mathbf{L}^2(T)$ such that $P \cup Q$ is linearly independent. Assume P is orthonormal. Then there exist barysymmetric $Q' \subset \mathbf{L}^2(T)$ such that $span(Q) = span(Q')$, and $P \perp Q'$.*

Proof. Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$. Define $Q' = \{q'_1, \ldots, q'_m\}$.

$$q'_j = q_j - \sum_{i=1}^{n} \langle q_j, p_i \rangle_T \, p_i \tag{4.24}$$

for $1 \leq j \leq m$. Then

1. Q' is a barysymmetric set and
2. $Q' \perp P$, which implies also $span(Q') = span(Q)$

as desired. □

Finally, we have,

Theorem 4.7 (S-orthogonalization) *Given a linearly independent and barysymmetric set $\mathcal{S}$, it is always possible to generate another barysymmetric set $\mathcal{S}'$ which is orthonormal and such that span($\mathcal{S}$) = span($\mathcal{S}'$).*

Proof. Define the following equivalence relation

$$p \sim_S q \Longleftrightarrow q \in orbit(p).$$

Then $\sim_S$ induces a partition on $\mathcal{S}$:

$$\mathcal{S} = \cup_{k=1,\text{disjoint}}^{N} orbit(p_k) \tag{4.25}$$

We proceed by induction on N. If N=1, then the result follows from Lemma 4.5. Assume the theorem is true for $N = N'$. When $N = N' + 1$, we can S-orthogonalize $\cup_{k=2}^{N'+1} orbit(p_k)$ into $\cup_{k=2}^{N'+1} orbit(p'_k)$ by the induction hypothesis. Then, by Lemma 4.6, we can S-orthogonalize $orbit(p_1)$ against $\cup_{k=2}^{N'+1} orbit(p'_k)$ and obtain p''_1 such that

$$orbit(p''_1) \perp \cup_{k=2}^{N'+1} orbit(p'_k).$$

Finally, we use again Lemma 4.5 to obtain p'_1 such that $orbit(p'_1)$ is orthonormal and

$$span(orbit(p'_1)) = span(orbit(p''_1)) = span(orbit(p_1)).$$

Then $\cup_{k=1}^{N'+1} orbit(p'_k)$ is an orthonormal set that has the same span as $\mathcal{S}$. □

The proof of Theorem 4.7 actually gives the following algorithm for S-orthogonalizing any given independent and barysymmetric set $\mathcal{S}$.

ALGORITHM: S-ORTHO

Description:

Symmetric orthogonalization of a given barysymmetric set.

Inputs:

$\mathcal{S}$, a linearly independent, barysymmetric set of functions

Outputs:

$\mathcal{S}'$, an orthonormal, barysymmetric set such that $span(S') = span(S)$

Algorithm

1. Initialize $i = 1$
2. Partition $\mathcal{S}$ into sets of *barysymmetric orbits* $\longrightarrow \mathcal{G} = \{G_1, G_2, \ldots, G_k\}$
3. S-orthogonalize $G_i \longrightarrow G'_i$
4. S-orthogonalize $G_{i+1}, \ldots, G_k$ with respect to G_i
5. $\mathcal{G} \longleftarrow \{G_{i+1}, \ldots, G_k\}$.
6. If $\mathcal{G}$ is empty, goto step 6, else set $i \longleftarrow i + 1$ and goto step 3.
7. Output $\mathcal{S}' = \{G'_1, G'_2, \ldots, G'_k\}$

We state and prove an additional result that will also be used in our second construction. We first need a new notation.

Let $P = \{p_1, \ldots, p_m\}, Q = \{q_1, \ldots, q_n\} \subset \mathbf{L}^2(T)$, we will denote by $\langle P,\ Q\rangle_T$ the m by n matrix

$$(\langle P,\ Q\rangle_T)_{i,j} = \langle p_i,\ q_j\rangle_T.$$

Definition 4.8 *We say Q is non-orthogonal to P if $\langle P,\ Q\rangle_T$ has full rank. We say Q is orthogonal to P if $P \subset Q^\perp$, or equivalently, $\langle P,\ Q\rangle_T = [0]_{|P|\times|Q|}$.*

Perhaps annoying, when P is non-orthogonal to Q in the above definition, it does not in general imply that P is "not orthogonal" to Q. Neither the reverse in true in general. The latter requires only $\langle P,\ Q\rangle_T$ has rank > 0, whereas the former is much more restrictive.

Theorem 4.9 *Given barysymmetric sets $P = orbit(p), Q = orbit(q)$ and R where $(P \cup Q \cup R)$ is linearly independent (i.e. $\dim(span(P \cup Q \cup R)) = |P| + |Q| + |R|$), Q is non-orthogonal to P and $|P| \leq |Q|$. We can always "S-orthogonalize R against P via Q", i.e. there exists barysymmetric set R' such that*

$$\begin{aligned} R' &\subset span(R \cup Q), \\ span(R \cup Q) &= span(R' \cup Q), \\ R' &\subset Q^{\perp}. \end{aligned} \tag{4.26}$$

Proof. $|P| \leq |Q|$ implies we have three distinct cases: (i) $|P| = |Q| = 1$, (ii) $|P| = 1$, $|Q| = 3$, (iii) $|P| = |Q| = 3$.

Partition R as in Equation 4.25, $R = \cup_{i=1}^{N} orbit(r_i)$.

In case (i), let

$$r_i' = r_i - \frac{\langle r_i, p\rangle}{\langle q, p\rangle} q,$$

and in case (ii), let $P = \{p_1, p_2, p_3\}$ and

$$r_i' = r_i - \frac{1}{3}\sum_{j=1}^{3} \frac{\langle r_i, p\rangle}{\langle q_j, p\rangle} q_j.$$

then it is clear that in both cases, $R' = \cup_{i=1}^{N} orbit(r_i')$ are as desired.

Now consider case(iii), we can think of P, Q and $R_i = orbit(r_i)$ as column 3-vectors. If $|orbit(r_i)| = 1$, simply make the three entries of R_i all equal to r_i. Notice that both $\langle Q, P\rangle_T$ and $\langle R, P\rangle_T$ are 3 by 3 *circulant* matrices. By assumption, $\langle Q, P\rangle_T$ is invertible. Therefore we can define

$$R_i' = R_i - \langle R_i,\ P\rangle_T \langle Q,\ P\rangle_T^{-1} Q. \tag{4.27}$$

Observe

1. $\langle Q,\ P\rangle_T^{-1}$ is also a circulant matrix, so both $\langle Q,\ P\rangle_T^{-1} Q$ and $\langle R_i,\ P\rangle_T \langle Q,\ P\rangle_T^{-1} Q$ constitute barysymmetric sets, hence so is R_i'.
2. if $|orbit(r_i)| = 1$, then all the nine entries of $\langle R_i,\ P\rangle_T$ are identically equal to $\langle r_i,\ p\rangle_T$, and all the three entries of R_i' are identical.

It should be clear now that $R' = \cup_{i=1}^{N} R_i'$ is as desired by the statement of the theorem. □

In fact the following more general statement is true, but we omit the proof because the underlying idea is similar to that of Theorem 4.9 but requires more clumsy notations. (After all, we don't need it for latter purposes.)

Theorem 4.10 *Given barysymmetric sets P, Q and R where $(P \cup Q \cup R)$ is linearly independent (i.e. $\dim(span(P \cup Q \cup R)) = |P| + |Q| + |R|$), $\langle Q\ ,\ P\rangle_T$ has full rank and $|P| \leq |Q|$. We can always "S-orthogonalize R against P via Q", i.e. there exists barysymmetric set R' such that*

1. $R' \subset span(R \cup Q)$,
2. $span(R \cup Q) = span(R' \cup Q)$,
3. $R' \subset Q^{\perp}$.

4.3 Construction II

We now construct multiwavelet bases for $\mathbf{L}^2(R)$

$$\{\beta_{T_{0,k}},\ g_{e^i_{j,k}} | i = 1, 2, 3,\ j \geq 0,\ k \in \mathcal{I}^j\}, \tag{4.28}$$

such that each $\beta_{T_{0,k}}$ or $g_{e^i_{j,k}}$ is a M-vector of functions that form a barysymmetric set with respect to the members' common support triangle. We will use extensively the tools established in the last section.

4.3.1 Scaling Functions

Given a triangle T, there is a basis for P_d or $P_d(T)$ which is barysymmetric (in fact strong-barysymmetric) with respect to T, namely, the *Bernstein basis* [2]

$$\{B^d_\lambda(\tau_1, \tau_2, \tau_3) = \tau_1^{\lambda_1}\tau_2^{\lambda_2}\tau_3^{\lambda_3} \mid |\lambda| = \sum_{i=1}^{3} \lambda_i = d\}. \tag{4.29}$$

The Bernstein basis is not an orthonormal basis for $P_d(T)$, and hence cannot be used directly as the scaling function of an orthonormal multiwavelet system. Theorem 4.7 says that there exists an orthogonalization of the Bernstein basis without destroying its barysymmetry property.

It is enough to say that the application of algorithm S-ORTHO to the Bernstein bases would generate the barysymmetric scaling functions for arbitrary d. There are however some specific knowledges of the Bernstein bases that can be used to facilitate the implementation.

First notice that $|orbit(B^d_\lambda)| = 1$ if and only if $\lambda_1 = \lambda_2 = \lambda_3$. Therefore, when d is indivisible by 3, the Bernstein basis of degree d is made up of orbits all with cardinality 3; and when d is divisible by 3, is made up of orbits all but one with cardinality 3, and that the singleton orbit is exactly $\{B^d_{d/3,d/3,d/3}\}$. Second, we have the following formula for $\mathcal{P}(B^d_\lambda)$ [15]:

$$\langle B^d_{(\lambda_1,\lambda_2,\lambda_3)}\ ,\ B^d_{(\lambda'_1,\lambda'_2,\lambda'_3)}\rangle_T = \frac{2|T|\ (\lambda_1+\lambda'_1)!\ (\lambda_2+\lambda'_2)!\ (\lambda_3+\lambda'_3)!}{(2d+2)!}$$

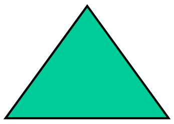
which implies,

$$\mathcal{P}(B^d_{(\lambda_1,\lambda_2,\lambda_3)}) = \langle B^d_{(\lambda_1,\lambda_2,\lambda_3)}\ ,\ B^d_{(\lambda_3,\lambda_1,\lambda_2)}\rangle_T = \frac{2|T|\ (\lambda_1+\lambda_2)!\ (\lambda_2+\lambda_3)!\ (\lambda_3+\lambda_1)!}{(2d+2)!} > 0.$$

Therefore, according to the working rule provided at the end of the proof of Lemma 4.5, we can always choose $r_3 = 1$ in the S-orthogonalization of any $orbit(B^d_\lambda)$. Figure 4 walks us through the construction of barysymmetric scaling functions in case of $d = 2$.

We will denote by $\{\mathcal{B}^l,\ 0 \leq l < M\}$ the S-orthogonalized Bernstein basis of degree $= d$, and $\beta^l_{T_{j,k}}$ the scaling functions defined in this section. Also, we will use the shorter term "O-Bernstein polynomials" to stand for the S-orthogonalized Bernstein polynomials.

[2] Traditionally, the Bernstein basis of degree d is defined as $B^d_\lambda(\tau_1,\tau_2,\tau_3) = \frac{d!}{\lambda_1!\,\lambda_2!\,\lambda_3!}\tau_1^{\lambda_1}\tau_2^{\lambda_2}\tau_3^{\lambda_3}$, which make the basis a partition of unity, but this property is of no relevance to our construction and we simply neglect the scalar coefficients and define the Bernstein basis as in Equation 4.29.



Bernstein Polynomial (d=2)

$\{\tau_1^{\lambda_1}\tau_2^{\lambda_2}\tau_3^{\lambda_3} \mid \lambda_1+\lambda_2+\lambda_3=2\}$

Split into barysymmetric orbits

$$\left\{\frac{\tau_1^2,\tau_2^2,\tau_3^2}{\tau_1\tau_2,\tau_2\tau_3,\tau_3\tau_1}\right\}$$

S − orthogonalize $\tau_1^2,\tau_2^2,\tau_3^2$

$$\left\{\frac{(a+b)\tau_1^2+(a-b)\tau_2^2+\tau_3^2,\tau_1^2+(a+b)\tau_2^2+(a-b)\tau_3^2,(a-b)\tau_1^2+\tau_2^2+(a+b)\tau_3^2,a=\frac{(-2-3\sqrt{10})}{4},b=\frac{-\sqrt{6}\sqrt{25-2\sqrt{10}}}{4}}{\tau_1\tau_2,\tau_2\tau_3,\tau_3\tau_1}\right\}$$

S - orthogonalize $\tau_1\tau_2,\tau_2\tau_3,\tau_3\tau_1$
w.r.t. $\tau_1^2,\tau_2^2,\tau_3^2$

$$\left\{\frac{(a+b)\tau_1^2+(a-b)\tau_2^2+\tau_3^2,\tau_1^2+(a+b)\tau_2^2+(a-b)\tau_3^2,(a-b)\tau_1^2+\tau_2^2+(a+b)\tau_3^2,a=\frac{(-2-3\sqrt{10})}{4},b=\frac{-\sqrt{6}\sqrt{25-2\sqrt{10}}}{4}}{\frac{(80\tau_1\tau_2-17\tau_1^2-17\tau_2^2-\tau_3^2)}{80},\frac{(80\tau_2\tau_3-\tau_1^2-17\tau_2^2-17\tau_3^2)}{80},\frac{(80\tau_3\tau_1-17\tau_1^2-\tau_2^2-17\tau_3^2)}{80}}\right\}$$

S - orthogonalize
$\frac{(80\tau_1\tau_2-17\tau_1^2-17\tau_2^2-\tau_3^2)}{80},\frac{(80\tau_2\tau_3-\tau_1^2-17\tau_2^2-17\tau_3^2)}{80},\frac{(80\tau_3\tau_1-17\tau_1^2-\tau_2^2-17\tau_3^2)}{80}$

$$\left\{\frac{-5.52\tau_1^2-.234\tau_2^2+\tau_3^2,-5.52\tau_2^2-.234\tau_3^2+\tau_1^2,-5.52\tau_3^2-.234\tau_1^2+\tau_2^2}{4.61\tau_1^2+4.36\tau_2^2-0.405\tau_3^2-22.8\tau_1\tau_2+2.25\tau_2\tau_3+\tau_3\tau_1,4.61\tau_2^2+4.36\tau_3^2-0.405\tau_1^2-22.8\tau_2\tau_3+2.25\tau_3\tau_1+\tau_1\tau_2,4.61\tau_3^2+4.36\tau_1^2-0.405\tau_2^2-22.8\tau_3\tau_1+2.25\tau_1\tau_2+\tau_2\tau_3}\right\}$$

Father Multiwavelets (d=2):
S-Orthogonalized Bernstein Polynomial

Figure 4: Scaling Functions in Construction II (d=2)

4.3.2 Mother

We now adapt the construction in Section 3.2.2 to construct barysymmetric multiwavelets, we will reuse all the notations there.

In this section we will generally denote by

$$\{g_i^l | i = 1, 2, 3, 0 \leq l < M\}$$

any *barysymmetric* set of $3M$ functions that can be combined with a basis of $P_d(T)$ to span $P_d(T_0) \oplus P_d(T_1) \oplus P_d(T_2) \oplus P_d(T_3)$

Let $\{\mathcal{B}_T^l\}$ be the O-Bernstein polynomials mutilated to the triangle T (c.f. Equation 3.11.)

Let

$$g_i^l(\tau_1, \tau_2, \tau_3) = \begin{cases} \mathcal{B}_{T_i}^l((\tau_1, \tau_2, \tau_3) M_{T \to T_i}) & \text{if } \tau_i \geq 1/2 \text{ and } (\tau_1, \tau_2, \tau_3) \geq 0, \\ -\mathcal{B}_{T_i}^l((\tau_1, \tau_2, \tau_3) M_{T \to T_i}) & \text{if } \tau_i < 1/2 \text{ and } (\tau_1, \tau_2, \tau_3) \geq 0 \\ 0 & \text{if } \tau_j < 0 \text{ for any } j \end{cases} \tag{4.30}$$

Notice that

1. $\{g_i^l\}$ is just like $\{H_i^l\}$ in Equation 3.15, except O-Bernstein bases are used instead of Legendre bases. Hence, $\{g_i^l\} \cup \{\mathcal{B}_T^l\}$ spans $P_d(T_0) \oplus P_d(T_1) \oplus P_d(T_2) \oplus P_d(T_3)$.

2. $orbit(g_1^l) = \{g_1^l, g_2^l, g_3^l\} := G^{l+1}$ and hence

$$\{g_i^l\} = \cup_{l=1}^M G^l.$$

   In particular, $\{g_i^l\}$ is a barysymmetric set.

Using equation 4.24 in Lemma 4.6, we S-orthogonalize each G^l against $\{\mathcal{B}_T^l\}$ and overwrite G^l by the result. Then $\cup_{l=0}^{M-1} G^l$ is now a basis for $\overline{W}$ which is also barysymmetric. As in construction I, we next arm $\cup_{l=0}^{M-1} G^l$ with more vanishing moments. In order to preserve barysymmetric, we have a do a vector version of the corresponding procedure in construction I. Assume we order the orbits of Bernstein polynomials on T of degree $> d$ as

$$\mathcal{O} = [\mathcal{O}_1, \mathcal{O}_2, \ldots]. \tag{4.31}$$

For example, if $d = 2$, then

$$\mathcal{O} = [\{\tau_1^3, \tau_2^3, \tau_3^3\}, \{\tau_1^2\tau_2, \tau_2^2\tau_3, \tau_3^2\tau_1\}, \{\tau_1\tau_2^2, \tau_2\tau_3^2, \tau_3\tau_1^2\}, \{\tau_1\tau_2\tau_3\}, \{\tau_1^4, \tau_2^4, \tau_3^4\}, \ldots]$$

.

We use the following procedure:

1. Initiatization: $\mathcal{L} = [G^1, G^2, \ldots, G^M]$, $t = 1$, $\mathcal{O}^{\text{target}} = \mathcal{O}_1$.

2. Find the first orbit, G, in the list $\mathcal{L}$which is non-orthogonal (in the sense of Definition 4.8.) We assume such a G is always available. (In fact, that is at the case from the actually calculation, at least for $d = 0, 1, 2$.)

3. reorder $\mathcal{L}$ as

$$\mathcal{L} \longleftarrow [G, [\mathcal{L} \backslash G]]$$

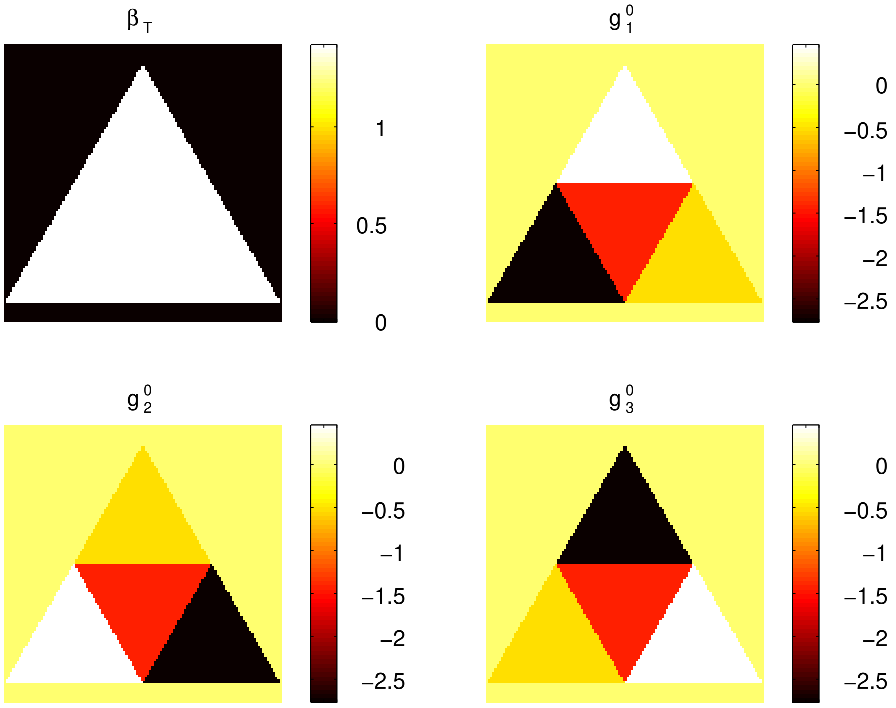


Figure 5: Barysymmetric Multiwavelets, $d = 0$

4. "Orthogonalize $[L \backslash G_1]$ against $\mathcal{O}^{\text{target}}$ via G_1" (c.f. Lemma 4.9)

5. $\mathcal{L} \longleftarrow [G_2, G_3, \ldots]$, $\mathcal{L}' \longleftarrow [\mathcal{L}', G_1]$. If $\mathcal{G}$ is now an empty list, $t \longleftarrow t+1$ goto 5, else goto 1.

6. $\mathcal{L} \longleftarrow \mathcal{L}'$

Note that $span(\mathcal{L})$ is unaffected and barysymmetry is preserved by the above operations, but that each G_{j+1} now has at least $M + \sum_{k=1}^{j} |\mathcal{O}^k|$ vanishing moments.

Finally, we apply step 2-6 of S-ORTHO to reverse($\mathcal{L}$) $= [G^M, G^{M-1}, \ldots, G^1]$.

Of course, we can mutilate the so constructed basis to arbitrary triangle $T_{j,k}$ using Equation 3.11.

Figure 5 and 6 displays the multiwavelets from construction II for $d = 0$ and 1 respectively.

4.3.3 Tradeoff between symmetry and vanishing moments

While the two constructions in Section 3.2.2 and Section 4.3.2 are similar in the overall spirit, the latter yields bases which are barysymmetric while the former yields bases with virtually no symmetry of any sort. On the other hand, the first construction yields bases of the form $\{h_1, h_2, \ldots, h_{3M}\}$ such that h_i has $M + i - 1$ vanishing moment. Therefore a so constructed basis has in average $(5M - 1)/2$ vanishing moments per element. The second construction yields bases $\{g_1, g_2, \ldots, g_{3M}\}$ such that $g_{3i+1}, g_{3i+2}, g_{3i+3}$ possess $M+3i$ vanishing moments, for $i = 0, 1, \ldots, (M-1)$. (We are assuming all the orbits in Equation 4.31 has cardinality 3, which means we are slightly overestimating the number of vanishing

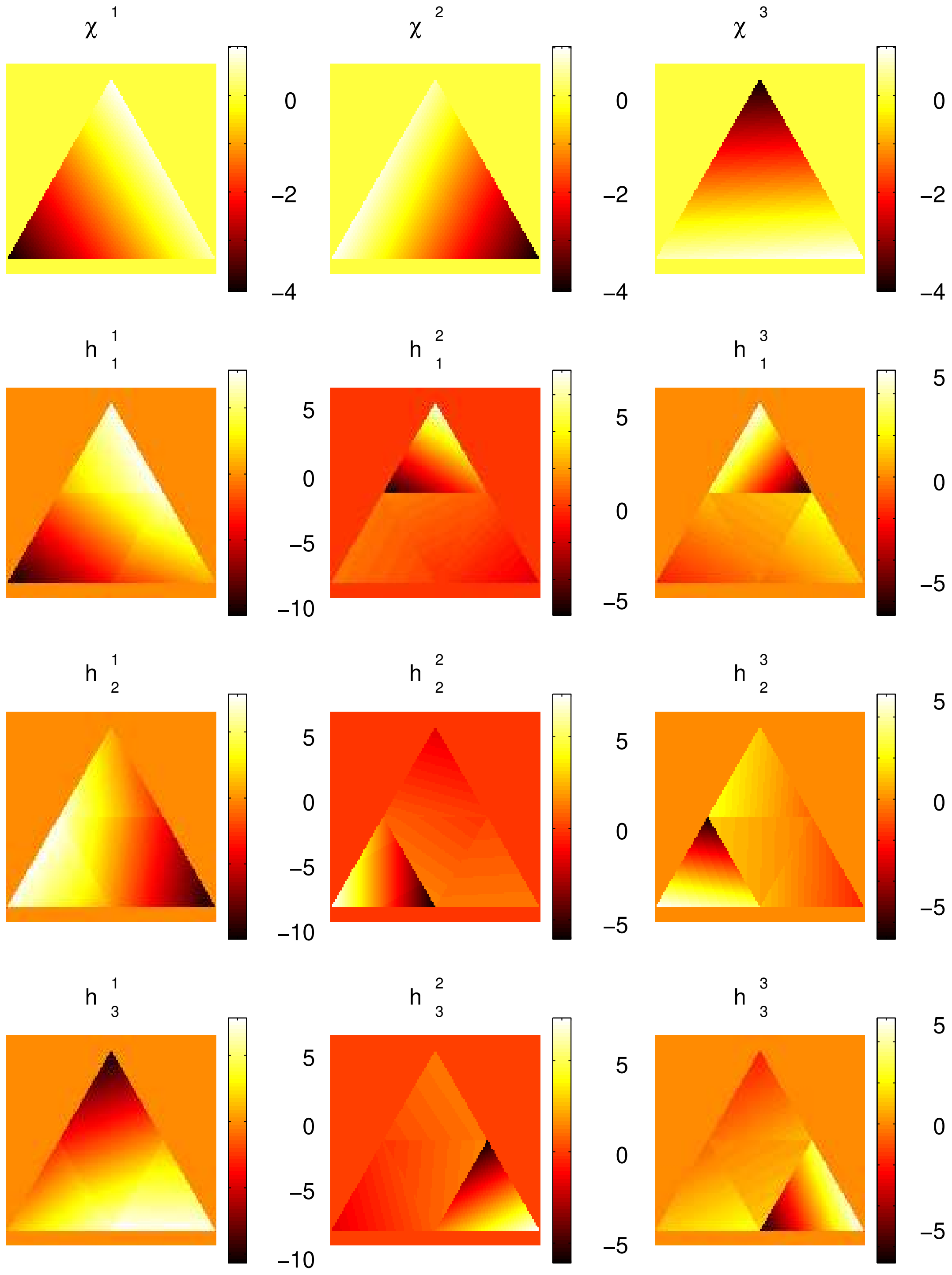


Figure 6: Barysymmetric Multiwavelets, $d = 1$

moments of the g_i's.) Thus such a basis has in average (approximately) $(5M-2)/6$ vanishing moments per element, which is about 3 times less than that in construction I.

In other words, there is a tradeoff in symmetry and vanishing moments. It occurs that such a tradeoff is necessary from the experience in wavelet design in 1-D. See, for example, Chapter 8 of [8].

It is also not hard to modify construction II in order to design multiwavelets that possess *strong-barysymmetry*, but that would involve sacrificing even more vanishing moments.

The relative importance of symmetry and vanishing moments occurs to be application-dependent. In applications where visual quality is involved symmetry would be important, whereas in numerical applications like those in [5] vanishing moment is all that matters.

5 Two-scale Relations, Filter Banks and Pyramid Algorithms

Given an arbitrary non-degenerate triangle T, and its midpoint subdivision $T = T_0 \cup T_1 \cup T_2 \cup T_3$. Construction I or II gives us an orthonormal basis $\{\chi_T^l | 0 \leq l < M-1\}$ for $P_d(T)$ and an orthonormal basis $\{h_i^l | i = 1,2,3, 0 \leq l < M-1\}$ for $\oplus_{i=0}^3 P_d(T_i) \ominus P_d(T)$. Therefore, there exists sixteen M by M matrices H_i, $G_{1,i}$, $G_{2,i}$, $G_{3,i}$, i=0,1,2,3, such that

$$\begin{aligned} \chi_T &= \sum_{i=0}^{3} H_i \, \chi_{T_i} \\ h_j &= \sum_{i=0}^{3} G_{j,i} \, \chi_{T_i} \text{ for } j = 1,2,3. \end{aligned}$$

By orthogonality, the filter banks $\{H_i, G_{k,i}\}$ can easily be calculated by

$$\begin{aligned} (H_i)_{p,q} &= \langle \chi_T^p \, , \, \chi_{T_i}^q \rangle \\ (G_{k,i})_{p,q} &= \langle h_k^p \, , \, \chi_{T_i}^q \rangle. \end{aligned} \tag{5.32}$$

Analogous to Equation 2, the $4M$ by $4M$ matrix

$$U = \begin{bmatrix} H_0 & H_1 & H_2 & H_3 \\ G_{1,0} & G_{1,1} & G_{1,2} & G_{1,3} \\ G_{2,0} & G_{2,1} & G_{2,2} & G_{2,3} \\ G_{3,0} & G_{3,1} & G_{3,2} & G_{3,3} \end{bmatrix} \tag{5.33}$$

is orthogonal. This is equivalent to saying we have two different orthonormal bases, $\{\chi_T^l, h_i^l\}$ and $\{\chi_{T_i}^l\}$ for $\oplus_{i=0}^3 P_d(T_i)$.

An important observation is that $\{H_i, G_{k,i}\}$ is completely independent of T. It can be seen easily from Equation 5.32 that the values of the integrals on the right hand sides are all independent of T.

6 Generalizations

6.1 Smooth Biorthogonal Duals

The multiwavelet bases constructed in this paper are discontinuous and orthonormal. It is possible to construct their smooth biorthogonal counterparts via moment-interpolation as described in [10].

6.2 Spherical Multiwavelets

Although we have been focusing on the planar case. The construction we have presented can potentially be extended to other surfaces. Notice that the basic components of constructions I and II are nested triangulation, barycentric coordinates and an inner-product.

We can, for instance, apply the above construction to construct multiwavelets on a sphere. A nested triangulation can be obtained by starting from an icosahedron and subdivide exactly as in the planar case, except triangles are now understood as geodesic triangles on the surface of the sphere. This type of nested triangulations of the sphere have been used, for example, by Swelden and Schröder in their construction of spherical wavelets [14]. Barycentric coordinates and Bernstein polynomials can also be defined on the sphere, see [1]. Of course, $\mathbf{L}^2(\mathbf{S}^2)$ is equipped with a well-defined inner-product.

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