1 Filtration

Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. Suppose that $\Sigma$ is a set of formulas closed under subformulas. We write say $w$ and $v$ are $\Sigma$-equivalent provided:

$$w \sim_{\Sigma} v \text{ iff for all } \varphi \in \Sigma, \mathcal{M}, w \models \varphi \iff \mathcal{M}, v \models \varphi.$$  

Note that $\sim_{\Sigma}$ is an equivalence relation. Let $|w|_{\Sigma} = \{v \mid w \sim_{\Sigma} v\}$ denote the equivalence class of $w$ under $\sim_{\Sigma}$.

Definition 1.1 (Filtration) Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. Given a set of formulas closed under subformulas, a model $\mathcal{M}^f = \langle W^f, R^f, V^f \rangle$ is said to be a filtration of $\mathcal{M}$ through $\Sigma$ provided

- $W^f = \{ |w|_{\Sigma} \mid w \in W \}$
- If $wRv$ then $|w|_{\Sigma}R^f|v|_{\Sigma}$
- If $|w|_{\Sigma}R^f|v|_{\Sigma}$ then for each $\Diamond \varphi \in \Sigma$, if $\mathcal{M}, v \models \varphi$ then $\mathcal{M}, w \models \Diamond \varphi$
- $V^f(p) = \{|w|_{\Sigma} \mid w \in V(p)\}$

Theorem 1.2 If $\mathcal{M}^f$ is a filtration of $\mathcal{M}$ through $\Sigma$, then for all $\varphi \in \Sigma$,

$$\mathcal{M}, w \models \varphi \iff \mathcal{M}^f, |w|_{\Sigma} \models \varphi.$$ 

Examples of Filtrations

- **smallest filtration**: $|w|_{\Sigma}R^s|v|_{\Sigma}$ iff there is $w' \in |w|_{\Sigma}$ and $v' \in |v|_{\Sigma}$ such that $w'Rv'$.
- **largest filtration**: $|w|_{\Sigma}R^l|v|_{\Sigma}$ iff for all $\Diamond \varphi \in \Sigma$, $\mathcal{M}, v \models \varphi$ implies $\mathcal{M}, w \models \Diamond \varphi$
- **transitive filtration**: $|w|_{\Sigma}R^t|v|_{\Sigma}$ iff for all $\Diamond \varphi \in \Sigma$, $\mathcal{M}, v \models \varphi \lor \Diamond \varphi$ implies $\mathcal{M}, w \models \Diamond \varphi$ (assuming $R$ is transitive)

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2 The Minimal Modal Logic

For a complete discussion of this material, consult Chapter 5 of Modal Logic for Open Minds by Johan van Benthem.

Definition 2.1 (Substitution) A substitution is a function from sentence letters to well formed modal formulas (i.e., $\sigma : At \to WFF_{ML}$). We extend a substitution $\sigma$ to all formulas $\varphi$ by recursion as follows (we write $\varphi^\sigma$ for $\sigma(\varphi)$):

1. $\sigma(\bot) = \bot$
2. $\sigma(\neg \varphi) = \neg \sigma(\varphi)$
3. $\sigma(\varphi \land \psi) = \sigma(\varphi) \land \sigma(\psi)$
4. $\sigma(\Box \varphi) = \Box \sigma(\varphi)$
5. $\sigma(\Diamond \varphi) = \Diamond \sigma(\varphi)$

For example, if $\sigma(p) = \Box \Diamond (p \land q)$ and $\sigma(q) = p \land \Box q$ then

$$(\Box (p \land q) \to \Box p)^\sigma = \Box ((\Box \Diamond (p \land q)) \land (p \land \Box q)) \to \Box (\Box \Diamond (p \land q))$$

Definition 2.2 (Tautology) A modal formula $\varphi$ is called a (propositional) tautology if $\varphi = (\alpha)^\sigma$ where $\sigma$ is a substitution, $\alpha$ is a formula of propositional logic and $\alpha$ is a tautology.

For example, $\Box p \to (\Diamond (p \land q) \to \Box p)$ is a tautology because $a \to (b \to a)$ is a tautology in the language of propositional logic and

$$(a \to (b \to a))^\sigma = \Box p \to (\Diamond (p \land q) \to \Box p)$$

where $\sigma(a) = \Box p$ and $\sigma(b) = \Diamond (p \land q)$.

Definition 2.3 (Modal Deduction) A modal deduction is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. $\alpha_i$ is a tautology
2. $\alpha_i$ is a substitution instance of $\Box (p \to q) \to (\Box p \to \Box q)$
3. $\alpha_i$ is of the form $\Box \alpha_j$ for some $j < i$
4. $\alpha_i$ follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that $\alpha_k$ is of the form $\alpha_j \to \alpha_i$).

We write $\vdash_K \varphi$ if there is a deduction containing $\varphi$.

The formula in item 2. above is called the $\textbf{K}$ axiom and the application of item 3. is called the rule of necessitation.

Fact 2.4 $\vdash_K \Box (\varphi \land \psi) \to (\Box \varphi \land \Box \psi)$
Proof.

1. $\phi \land \psi \rightarrow \phi$ \hspace{1cm} tautology
2. $\Box((\phi \land \psi) \rightarrow \phi)$ \hspace{1cm} Necessitation 1
3. $\Box((\phi \land \psi) \rightarrow \phi) \rightarrow (\Box(\phi \land \psi) \rightarrow \Box \phi)$ \hspace{1cm} Substitution instance of K
4. $\Box(\phi \land \psi) \rightarrow \Box \phi$ \hspace{1cm} MP 2,3
5. $\phi \land \psi \rightarrow \psi$ \hspace{1cm} tautology
6. $\Box((\phi \land \psi) \rightarrow \psi)$ \hspace{1cm} Necessitation 5
7. $\Box((\phi \land \psi) \rightarrow \psi) \rightarrow (\Box(\phi \land \psi) \rightarrow \Box \psi)$ \hspace{1cm} Substitution instance of K
8. $\Box(\phi \land \psi) \rightarrow \Box \psi$ \hspace{1cm} MP 5,6
9. $(a \rightarrow b) \rightarrow ((a \rightarrow c) \rightarrow (a \rightarrow (b \land c)))$ \hspace{1cm} tautology ($a := \Box(\phi \land \psi), b := \Box \phi, c := \Box \psi$)
10. $(a \rightarrow c) \rightarrow (a \rightarrow (b \land c))$ \hspace{1cm} MP 4,9
11. $\Box(\phi \land \psi) \rightarrow \Box \psi \land \Box \psi$ \hspace{1cm} MP 8,10

QED

Fact 2.5 If $\vdash_{K} \phi \rightarrow \psi$ then $\vdash_{K} \Box \phi \rightarrow \Box \psi$

Proof.

1. $\phi \rightarrow \psi$ \hspace{1cm} assumption
2. $\Box(\phi \rightarrow \psi)$ \hspace{1cm} Necessitation 1
3. $\Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$ \hspace{1cm} Substitution instance of K
4. $\Box \phi \rightarrow \Box \psi$ \hspace{1cm} MP 2,3

QED

Definition 2.6 (Modal Deduction with Assumptions) Let $\Sigma$ be a set of modal formulas. A modal deduction of $\phi$ from $\Sigma$, denoted $\Sigma \vdash_{K} \phi$ is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ where for each $i \leq n$ either

1. $\alpha_i$ is a tautology
2. $\alpha_i \in \Sigma$
3. $\alpha_i$ is a substitution instance of $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
4. $\alpha_i$ is of the form $\Box \alpha_j$ for some $j < i$ and $\vdash_{K} \alpha_j$
5. $\alpha_i$ follows by modus ponens from earlier formulas (i.e., there is $j, k < i$ such that $\alpha_k$ is of the form $\alpha_j \rightarrow \alpha_i$).

Remark 2.7 Note that the side condition in item 4. in the above definition is crucial. Without it, one application of Necessitation shows that $\{p\} \vdash_{K} \Box p$. Using the general fact (cf. Exercise #4, Section 1.2 of Enderton) that $\Sigma; \alpha \vdash_{K} \beta$ imply $\Sigma \vdash_{K} \alpha \rightarrow \beta$, we can conclude that $\vdash_{K} p \rightarrow \Box p$. But, clearly $p \rightarrow \Box p$ cannot be a theorem (why?).

Definition 2.8 (Logical Consequence) Suppose that $\Sigma$ is a set of modal formulas. We say $\phi$ is a logical consequence of $\Sigma$, denoted $\Sigma \models \phi$ provided for all frames $\mathcal{F}$, if $\mathcal{F} \models \alpha$ for each $\alpha \in \Sigma$, then $\mathcal{F} \models \phi$.  

$\diamondsuit$
Theorem 2.9 (Soundness) If $\Sigma \vdash K \varphi$ then $\Sigma \models \varphi$.

Proof. The proof is by induction on the length of derivations. See Chapter 5 in Modal Logic for Open Minds and your lecture notes. QED

Theorem 2.10 (Completeness) If $\Sigma \models \varphi$ then $\Sigma \vdash K \varphi$.

Proof. See Chapter 5 in Modal Logic for Open Minds and your lecture notes for a proof. QED

Remark 2.11 (Alternative Statement of Soundness and Completeness) Suppose that $\Sigma$ is a set of modal formulas. Define the minimal modal logic as the smallest set $\Lambda_K(\Sigma)$ of modal formulas extending $\Sigma$ that (1) contains all tautologies, (2) contains the formula $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, (3) is closed under substitutions, (4) is closed under the Necessitation rule (i.e., if $\varphi \in \Lambda_K$ is derivable without premises $\vdash K \varphi$ then $\Box \varphi \in \Lambda_K$) and (4) is closed under Modus Ponens. Suppose $\mathcal{F}(\Sigma) = \{ \varphi | \Sigma \models \varphi \}$. Then, soundness and completeness states that $\Lambda_K(\Sigma) = \mathcal{F}(\Sigma)$.

Some Axioms Some Modal Logics

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<thead>
<tr>
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<tbody>
<tr>
<td>$K$</td>
<td>$\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$</td>
<td>$K$</td>
<td>$K + PC + \text{Nec}$</td>
</tr>
<tr>
<td>$D$</td>
<td>$\Box \varphi \rightarrow \Diamond \varphi$</td>
<td>$T$</td>
<td>$K + T + PC + \text{Nec}$</td>
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<tr>
<td>$T$</td>
<td>$\Box \varphi \rightarrow \varphi$</td>
<td>$S4$</td>
<td>$K + T + 4 + PC + \text{Nec}$</td>
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<tr>
<td>$4$</td>
<td>$\Box \varphi \rightarrow \Box \Box \varphi$</td>
<td>$S5$</td>
<td>$K + T + 4 + 5 + PC + \text{Nec}$</td>
</tr>
<tr>
<td>$5$</td>
<td>$\neg \Box \varphi \rightarrow \Box \neg \Box \varphi$</td>
<td>$K\text{D}4\text{S}$</td>
<td>$K + D + 4 + 5 + PC + \text{Nec}$</td>
</tr>
<tr>
<td>$W$</td>
<td>$\Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi$</td>
<td>$\text{GL}$</td>
<td>$K + W$</td>
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Completeness Theorems

- $T$ is sound and strongly complete with respect to the class reflexive Kripke frames.
- $S4$ is sound and strongly complete with respect to the class reflexive Kripke frames.
- $S5$ is sound and strongly complete with respect to the class reflexive Kripke frames.
- $K\text{D}4\text{S}5$ is sound and strongly complete with respect to the class reflexive Kripke frames.

3 Alternative Proof of Weak Completeness

In this section we illustrate a technique for by proving weak completeness invented by Larry Moss in [1]. Since we are only interested in illustrating the technique, we focus on the smallest normal modal logic ($K$). Recall that the basic modal language is generated by the following grammar:

$$p \mid \neg \varphi \mid \varphi \land \psi \mid \Diamond \varphi$$

where $p$ is a propositional variable (let $At = \{p_1, p_2, \ldots, p_n, \ldots\}$ denote the set of propositional variables). Define the usual boolean connectives and the modal operator $\Box$ as usual. Let $L_\Box$ be the set of well-formed formulas.

Some notation is useful at this stage. The height, or modal depth, of a formula $\varphi \in L_\Box$, denoted $ht(\varphi)$, is longest sequence of nested modal operators. Formally, define $ht$ as follows...
The order of a modal formula \( \varphi \), written \( \text{ord}(\varphi) \), is the largest index of a propositional formula that appears in \( \varphi \). Formally,

\[
\begin{align*}
\text{ord}(p_n) &= n \\
\text{ord}(\neg \varphi) &= \text{ord}(\varphi) \\
\text{ord}(\varphi \lor \psi) &= \max\{\text{ord}(\varphi), \text{ord}(\psi)\} \\
\text{ord}(\Box \varphi) &= 1 + \text{ord}(\varphi)
\end{align*}
\]

Let \( \mathcal{L}_{h,n} = \{ \varphi \mid \varphi \in \mathcal{L}_o, \text{ht}(\varphi) \leq h \text{ and } \text{ord}(\varphi) \leq n \} \). Thus, for example, \( \mathcal{L}_{0,n} \) is the propositional language (finite up to logical equivalence) built from the set \( \{p_1, \ldots, p_n\} \) of propositional variables.

A set \( T \subseteq \{p_1, \ldots, p_m\} \) corresponds to a partial valuation on \( \mathcal{L} \) if we think of the elements of \( T \) as being true and the elements of \( \{p_1, \ldots, p_m\} - T \) as being false. This partial valuation can be described by the following formula of \( \mathcal{L}_{0,m} \)

\[
\hat{T} = \bigwedge_{p \in T} p \land \bigwedge_{p \in \{p_1, \ldots, p_n\} - T} \neg p
\]

Now, for each \( \varphi \in \mathcal{L}_{0,m} \) it is easy to see that exactly one of the following holds: \( \vdash \hat{T} \rightarrow \varphi \) or \( \vdash \hat{T} \rightarrow \neg \varphi \). Furthermore, it is easy to show that for each \( \varphi \in \mathcal{L}_{0,m}, \vdash \varphi \leftrightarrow \bigvee \{\hat{T} \mid \vdash \hat{T} \rightarrow \varphi\} \). The central idea of Moss’ technique is to generalize these facts to modal logic.

It is well-known that modal logic has the finite tree property, i.e., when evaluating a formula \( \varphi \) it is enough to consider only paths of length at most the modal depth of \( \varphi \). The modal generalization of the formulas described above are called canonical sentences. Fix a natural number \( n \) and construct a set of canonical sentences, denoted \( \mathcal{C}_{h,n} \), by induction on \( h \). Let \( \mathcal{C}_{0,n} = \{ \hat{T} \mid T \subseteq \{p_1, \ldots, p_n\} \} \). Suppose that \( \mathcal{C}_{h,n} \) has been defined and that \( S \subseteq \mathcal{C}_{h,n} \) and \( T \subseteq \{p_1, \ldots, p_n\} \). Define the formula

\[
\alpha_{S,T} := \bigwedge_{\psi \in S} \Box \psi \land \bigvee S \land \hat{T}
\]

and let \( \mathcal{C}_{h+1,n} = \{ \alpha_{S,T} \mid S \subseteq \mathcal{C}_{h,n}, T \subseteq \{p_1, \ldots, p_n\} \} \). It is not hard to see that formulas of the form \( \alpha_{S,T} \) play the same role in modal logic as the formulas \( \hat{T} \) in propositional logic. That is, \( \alpha_{S,T} \) can be thought of as a complete description of a modal state of affairs. This is justified by the following Lemma from [1]. The proof can be found in [1] although we will repeat it here in the interest of exposition.

**Lemma 3.1** For any modal formula \( \varphi \) of modal depth at most \( h \) built from propositional variables \( \{p_1, \ldots, p_n\} \) and any \( \alpha_{S,T} \in \mathcal{C}_{h+1,n} \) exactly one of the following holds \( \vdash \alpha_{S,T} \rightarrow \varphi \) or \( \vdash \alpha_{S,T} \rightarrow \neg \varphi \).

**Proof.** The proof is by induction on \( \varphi \). The base case is obvious as are the boolean connectives. We consider only the modal case. Suppose that statement holds for \( \psi \) and consider the formula \( \Diamond \psi \). Note that for each \( \beta \in S \), the induction hypothesis applies to \( \beta \) and \( \psi \). Thus for each \( \beta \in S \), either \( \vdash \beta \rightarrow \psi \) or \( \vdash \beta \rightarrow \neg \psi \). There are two cases: 1. there is some \( \beta \in S \) such that \( \vdash \beta \rightarrow \psi \) and 2. for each \( \beta \in S \), \( \vdash \beta \rightarrow \neg \psi \). Suppose case 1 holds and \( \beta \in S \) is such that \( \vdash \beta \rightarrow \psi \). Then, it
is easy to show that in $K$, $\vdash \Diamond \beta \to \Diamond \psi$. Hence, by construction of $\alpha_{S,T}$, $\vdash \alpha_{S,T} \to \Diamond \psi$. Suppose we are in the second case. Using propositional reasoning, $\vdash \lor S \to \neg \psi$. Then, $\vdash \Box \lor S \to \Box \neg \psi$. Hence, by construction of $\alpha_{S,T}$, $\vdash \alpha_{S,T} \to \neg \Diamond \psi$.

This lemma demonstrates that we can think of these formulas as complete descriptions of a state (up to finite depth) in some Kripke structure. There are a few other facts that are relevant at this point. The proofs can be found in [1] and we will not repeat them here. Given a set of formulas $X$, let $\bigoplus X$ denote exactly one of $X$. Formally, if $X = \{\varphi_1, \ldots, \varphi_n\}$, then $\bigoplus X$ is short for $\lor_{i=1, \ldots, n} (\varphi_i \land \neg \lor_{j \neq i} \varphi_j)$.

**Lemma 3.2** 1. For any $h$, $\vdash \bigoplus C_{h,n}$ (and hence $\vdash \lor C_{h,n}$)

2. For any formula $\varphi$ of height $h$, $\vdash \varphi \leftrightarrow \lor \{\alpha \mid \alpha \in C_{h,n}, \vdash \alpha \rightarrow \varphi\}$

Moss constructs a (finite) Kripke model from the set of formulas $C_{h,n}$ as follows. Let $C_{h,n} = \langle C, R, V \rangle$ where

1. $C \subseteq C_{h,n}$ is the set of all $K$-consistent formulas from $C_{h,n}$

2. For $\alpha, \beta \in C$, $\alpha R \beta$ provided $\alpha \land \Diamond \beta$ is consistent

3. for $p \in \{p_1, \ldots, p_n\}$, $V(p) = \{\alpha \mid \alpha \in C, \vdash \alpha \rightarrow p\}$.

The truth Lemma connects truth of $\varphi$ at a state $\alpha$ and the derivability of the implication $\alpha \rightarrow \varphi$. We first need an existence Lemma whose proof can be found in [1].

**Lemma 3.3 (Existence Lemma, [1])** Suppose that $\varphi \in L_{h,n}$ and $C_{h,n} = \langle C, R, V \rangle$ is as defined above. If $\alpha \land \Diamond \varphi$ is $K$-consistent then there is a $\beta \in C$ such that $\alpha \land \Diamond \beta$ is $K$-consistent and $\vdash \beta \rightarrow \varphi$.

The proof uses Lemma 3.2 and can be found in [1].

**Lemma 3.4 (Truth Lemma, [1])** Suppose that $\varphi \in L_{h,n}$ and $C_{h,n} = \langle C, R, V \rangle$ is as defined above. Then for each $\alpha \in C$, $C_{h,n}, \alpha \models \varphi$ iff $\vdash_K \alpha \rightarrow \varphi$.

**Proof.** As usual, the proof is by induction on $\varphi$. The base case and boolean connectives are straightforward. The only interesting case is the modal operator. Suppose that $C_{h,n}, \alpha \models \Diamond \psi$. Then there is some $\beta \in C$ such that $\alpha R \beta$ and $C_{h,n}, \beta \models \psi$. By the definition of $R$, $\alpha \land \Diamond \beta$ is $K$-consistent. By Lemma 3.1, either $\vdash \alpha \rightarrow \Diamond \psi$ or $\vdash \alpha \rightarrow \neg \Diamond \psi$. If $\vdash \alpha \rightarrow \Diamond \psi$ we are done. Suppose that $\vdash \alpha \rightarrow \neg \Diamond \psi$. Now, by the induction hypothesis, $\vdash \beta \rightarrow \psi$. Hence $\vdash \Diamond \beta \rightarrow \Diamond \psi$. But this contradicts the assumption that $\alpha \land \Diamond \beta$ is $K$-consistent. Suppose that $\vdash \alpha \rightarrow \Diamond \psi$. Then $\alpha \land \Diamond \psi$ is $K$-consistent. Hence by Lemma 3.3, there is a $\beta \in C$ such that $\alpha \land \Diamond \beta$ is $K$-consistent and $\vdash \beta \rightarrow \psi$. But this means that $C_{h,n}, \alpha \models \Diamond \psi$.

The weak completeness theorem easily follows from the above Lemmas.

**Theorem 3.5** $K$ is weakly complete, i.e., for each $\varphi \in L_o$, if $\models \varphi$, then $\vdash_K \varphi$. 

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**Proof.** Let \( h \) and \( n \) be large enough so that \( \varphi \in L_{h,n} \) and suppose that \( \models \varphi \). Then, in particular, \( \varphi \) is valid in \( C_{h,n} \). Thus for each \( \alpha \in C \), \( C_{h,n}, \alpha \models \varphi \). Hence by Lemma 3.4, for each \( \alpha \in C \), \( \vdash \alpha \rightarrow \varphi \). Hence, \( \vdash \bigvee C \rightarrow \varphi \). By Lemma 3.2, \( \vdash \bigvee C \). Therefore, \( \vdash \varphi \).  

In [1], Moss uses the above technique to show that a number of well-known modal logics are weakly complete.

**References**