Modal Logic
Introductory Lecture

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January 31, 2012
Modern Modal Logic began with C.I. Lewis’ dissatisfaction with the material conditional ($\rightarrow$).
Dorothy Edgington’s Proof of the Existence of God

\[ \neg G \rightarrow \neg (P \rightarrow A) \]

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If God does not exist, then it’s not the case that if I pray, my prayers will be answered.
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I don’t pray
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$\neg G \rightarrow \neg (P \rightarrow A)$

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\[
\begin{array}{ccc}
\neg G & \neg (P \rightarrow A) & \neg G \rightarrow \neg (P \rightarrow A) \\
T & T & T \\
T & F & F \\
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\end{array}
\]

\[
\neg G \rightarrow \neg (P \rightarrow A) \\
\neg P \\
G
\]

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If God does not exist, then it’s not the case that if I pray, my prayers will be answered.

I don’t pray \[ \neg P \]

\[ \neg G \rightarrow \neg (P \rightarrow A) \]

Therefore, God exists!
Setting the Stage

C.I. Lewis’ idea: Interpret ‘If $A$ then $B$’ as ‘It must be the case that $A$ implies $B$’, or ‘It is necessarily the case that $A$ implies $B$’
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Prosecutor: $G \rightarrow A$
Defense: $\neg(G \rightarrow A)$
Judge: $\neg(G \rightarrow A)$
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Defense: $\neg (G \rightarrow A)$
Judge: $\neg (G \rightarrow A) \iff G \land \neg A$, therefore $G$!
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Prosecutor: $\Box(G \rightarrow A)$ (It is necessarily the case that . . . )
Defense: $\neg\Box(G \rightarrow A)$
Judge: $\neg\Box(G \rightarrow A)$ (What can the Judge conclude?)
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Gradually, the study of the modalities themselves became dominant, with the study of “implication” developing into a separate topic.
Setting the Stage

\( \square \varphi \): “It is \textit{necessarily} the case that \( \varphi \)” (“It must be that \( \varphi \)”)  
\( \diamond \varphi \): “It is \textit{possible} that \( \varphi \)” (“It can/might be that \( \varphi \)”)
Setting the Stage: Different Senses of “Possibility”

- I can come to the party, but I can’t stay late. (“is not inconvenient”)

- Humans can travel to the moon, but not Mars. (“is achievable with current technology”)

- It’s possible to move almost as fast as the speed of light, but not to travel faster than light. (“is consistent with the laws of nature”)

- Objects could have traveled faster than the speed of light (if the laws of nature had been different), but no matter what the laws had been, nothing could have traveled faster than itself. (“metaphysical possibility”)

- You may borrow but you may not steal. (“is morally acceptable”)

- It might rain tomorrow. (“is epistemic possibility”)

Modal Logic 6/45
Setting the Stage: Different Senses of “Possibility”

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Modal Logic 6/45
The History of Modal Logic


What is a modal?

A **modality** is any word or phrase that can be applied to a given states $S$ to create a new statement that makes an assertion about the mode of truth of $S$. 
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John _______ happy.
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- ...
Types of Modal Logics

tense: henceforth, eventually, hitherto, deviously, now, tomorrow, yesterday, since, until, inevitably, finally, ultimately, endlessly, it will have been, it is being, ...
Types of Modal Logics

**tense:** henceforth, eventually, hitherto, deviously, now, tomorrow, yesterday, since, until, inevitably, finally, ultimately, endlessly, it will have been, it is being, . . .

**epistemic:** it is known to a that, it is common knowledge that
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**dynamic**: after the program/computation/action finishes, the program enables, throughout the computation
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**geometric**: it is locally the case that
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deontic: it is obligatory/forbidden/permitted/unlawful that

dynamic: after the program/computation/action finishes, the program enables, throughout the computation

geometric: it is locally the case that

metalogic: it is valid/satisfiable/provable/consistent that
A formula of Modal Logic is defined *inductively*:

1. Any atomic propositional variable is a formula
2. If $P$ and $Q$ are formula, then so are $\neg P$, $P \land Q$, $P \lor Q$ and $P \rightarrow Q$
3. If $P$ is a formula, then so is $\square P$ and $\diamond P$
The Basic Modal Language

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Boolean Logic
The Basic Modal Language

A formula of Modal Logic is defined *inductively*:

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**Unary operator**
The Basic Modal Language

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3. If $P$ is a formula, then so is $\Box P$ and $\Diamond P$

Eg., $\Box(P \rightarrow \Diamond Q) \lor \Box \Diamond \neg R$
Modal Formulas: $\neg \Box \varphi \rightarrow \psi$

$\neg (\Box \varphi \rightarrow \psi)$  $\neg \Box (\varphi \rightarrow \psi)$  $\neg \Box \varphi \rightarrow \psi$
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Narrow vs. Wide Scope

“If you do $p$, you must also do $q$”

- $p \rightarrow \Box q$
- $\Box(p \rightarrow q)$
Narrow vs. Wide Scope

“If you do $p$, you must also do $q$”

- $p \rightarrow \square q$
- $\square(p \rightarrow q)$

“If Bob is a bachelor, then he is necessarily unmarried”

- $B \rightarrow \square U$
- $\square(B \rightarrow U)$
de dicto vs. de re

“I know that someone appreciates me”

- □∃xA(x, e) (de dicto)
- ∃x□A(x, e) (de re)
Iterations of Modal Operators

□φ → □□φ: If I know, do I know that I know?

¬□φ → □¬□φ: If I don’t know, do I know that I don’t know?
Modal reasoning patterns
Formal modeling
Deontic Logic

$OA$ means $A$ is obligatory

$PA$ means $A$ is permitted
Deontic Logic

$OA$ means $A$ is obligatory
$PA$ means $A$ is permitted

Is the following argument valid?

If $A$ then $B$ ($A \rightarrow B$)

If $A$ is obligatory then so is $B$ ($OA \rightarrow OB$)
1. Jones murders Smith. \((M)\)
2. If Jones murders Smith, then Jones ought to murder Smith gently. \((M \rightarrow OG)\)

(first discussed by J. Forrester in 1984)
Deontic Logic

1. Jones murders Smith. \((M)\)
2. If Jones murders Smith, then Jones ought to murder Smith gently. \((M \rightarrow OG)\)

? Jones ought to murder Smith. \((OM)\)

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Deontic Logic

✓ Jones murders Smith. (M)
✓ If Jones murders Smith, then Jones ought to murder Smith gently. (M → OG)

3. Jones ought to murder Smith gently. (OG)

? Jones ought to murder Smith. (OM)

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1. Jones murders Smith. \((M)\)
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\(\Rightarrow\) If Jones murders Smith gently, then Jones murders Smith. \((G \rightarrow M)\)

? Jones ought to murder Smith. \((OM)\)

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Deontic Logic

1. Jones murders Smith. \((M)\)
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3. Jones ought to murder Smith gently. \((OG)\)
   ✓ If Jones murders Smith gently, then Jones murders Smith. \((G \rightarrow M)\)

(Mon) If Jones ought to murder Smith gently, then Jones ought to murder Smith. \((OG \rightarrow OM)\)

? Jones ought to murder Smith. \((OM)\)

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✓ Jones ought to murder Smith gently. \((OG)\)

4. If Jones murders Smith gently, then Jones murders Smith. \((G \rightarrow M)\)

✓ If Jones ought to murder Smith gently, then Jones ought to murder Smith. \((OG \rightarrow OM)\)

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4. If Jones murders Smith gently, then Jones murders Smith. \((G \rightarrow M)\)
5. If Jones ought to murder Smith gently, then Jones ought to murder Smith. \((OG \rightarrow OM)\)
6. Jones ought to murder Smith. \((OM)\)

(first discussed by J. Forrester in 1984)
THREE LOGICIANS WALK INTO A BAR...

DOES EVERYONE WANT BEER?

I DON'T KNOW.

I DON'T KNOW.

YES!

spikedmath.com
© 2011
States
States

Logician 1 wants a beer

Modal Logic
States

Logician 1 wants a beer

Logician 2 does not want a beer
States

Logician 1 wants a beer

Logician 2 does not want a beer

Logician 3 wants a beer
A Model of the Logicians’ Information
A Model of the Logicians’ Information
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A Model of the Logicians’ Information
Logician 1: “I don’t know”
Logician 1: “I don’t know”
Logician 1: “I don’t know”
Logician 2: “I don’t know”
Logician 2: “I don’t know”
Logician 3: “Yes!”
Actions

1. Actions as *transitions between states, or situations*:
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\[ s \xrightarrow{a} t \]
Actions

1. Actions as *transitions between states, or situations*:

2. Actions *restrict* the set of possible future histories.
It turns out that structured actions can be viewed as compositions of basic actions, with only a few basic composition recipes: conditional execution, choice, sequence, and repetition. In some cases it is also possible to undo or reverse an action. This gives a further recipe: if you are editing a file, you can undo the last 'delete word' action, but you cannot undo the printing of your file.

Conditional or guarded execution (“remove from fire when cheese starts to melt”), sequence (“pour eggs in and swirl; cook for about three minutes; gently slide out of the pan”), and repetition (“keep stirring until soft”) are ways in which a cook combines his basic actions in preparing a meal. But these are also the strategies for a lawyer when planning her defence (“only discuss the character of the defendant if the prosecution forces us”, “first convince the jury of the soundness of the alibi, next cast doubt on the reliability of the witness for the prosecution”), or the basic layout strategies for a programmer in designing his code. In this chapter we will look at the logic of these ways of combining actions.

Action structure does not depend on the nature of the basic actions: it applies to actions in the world, such as preparing breakfast, cleaning dishes, or spilling coffee over your trousers. It also applies to communicative actions, such as reading an English sentence and updating one’s state of knowledge accordingly, engaging in a conversation, sending an email with cc’s, telling your partner a secret. These actions typically change the cognitive states of the agents involved. Finally, it applies to computations, i.e., actions performed by computers. Examples are computing the factorial function, computing square roots, etc. Such actions typically involve changing the memory state of a machine. Of course there are connections between these categories. A communicative action will usually involve some computation involving memory, and the utterance of an imperative (‘Shut the door!’) is a communicative action that is directed towards action in the world.

There is a very general way to model action and change, a way that we have in fact seen already. The key is to view a changing world as a set of situations linked by labeled arcs. In the context of epistemic logic we have looked at a special case of this, the case where the arcs are epistemic accessibility relations: agent relations that are reflexive, symmetric, and transitive. Here we drop this restriction.

Consider an action that can be performed in only one possible way. Toggling a switch for switching off your alarm clock is an example. This can be pictured as a transition from an initial situation to a new situation:

```
alarm on
```

Toggling the switch once more will put the alarm back on:

```
toggle
alarm off
```

Some actions do not have determinate effects. Asking your boss for a promotion may get you promoted, but it may also get you fired, so this action can be pictured like this:

```
employed
promoted
fired
ask for promotion
```

Another example: opening a window. This brings about a change in the world, as follows.

```
open window
```

The action of window-opening changes a state in which the window is closed into one in which it is open. This is more subtle than toggling an alarm clock, for once the window is open a different action is needed to close it again. Also, the action of opening a window can only be applied to closed windows, not to open ones. We say: performing the action has a precondition or presupposition.

In fact, the public announcements from the previous chapter can also be viewed as (communicative) actions covered by our general framework. A public announcement is an action that effects a change in an information model.
analyzing the interplay of action and static descriptions of the world before and after the action. It turns out that structured actions can be viewed as compositions of basic actions, with only a few basic composition recipes: conditional execution, choice, sequence, and repetition. In some cases it is also possible to undo or reverse an action. This gives a further recipe: if you are editing a file, you can undo the last 'delete word' action, but you cannot undo the printing of your file.

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Action structure does not depend on the nature of the basic actions: it applies to actions in the world, such as preparing breakfast, cleaning dishes, or spilling coffee over your trousers. It also applies to communicative actions, such as reading an English sentence and updating one's state of knowledge accordingly, engaging in a conversation, sending an email with cc's, telling your partner a secret. These actions typically change the cognitive states of the agents involved. Finally, it applies to computations, i.e., actions performed by computers. Examples are computing the factorial function, computing square roots, etc. Such actions typically involve changing the memory state of a machine. Of course there are connections between these categories. A communicative action will usually involve some computation involving memory, and the utterance of an imperative ("Shut the door!") is a communicative action that is directed towards action in the world.

There is a very general way to model action and change, a way that we have in fact seen already. The key is to view a changing world as a set of situations linked by labeled arcs. In the context of epistemic logic we have looked at a special case of this, the case where the arcs are epistemic accessibility relations: agent relations that are reflexive, symmetric, and transitive. Here we drop this restriction.

Consider an action that can be performed in only one possible way. Toggling a switch for switching off your alarm clock is an example. This can be pictured as a transition from an initial situation to a new situation:

```
alarm on  toggle  alarm off
```

Toggling the switch once more will put the alarm back on:

```
alarm on  toggle  alarm on
```

Some actions do not have determinate effects. Asking your boss for a promotion may get you promoted, but it may also get you fired, so this action can be pictured like this:

```
employed  ask for promotion  promoted
```

Another example: opening a window. This brings about a change in the world, as follows.

```
open window
```

The action of window-opening changes a state in which the window is closed into one in which it is open. This is more subtle than toggling an alarm clock, for once the window is open a different action is needed to close it again. Also, the action of opening a window can only be applied to closed windows, not to open ones. We say: performing the action has a precondition or presupposition.

In fact, the public announcements from the previous chapter can also be viewed as (communicative) actions covered by our general framework. A public announcement is an action that effects a change in an information model.
6.1. ACTIONS IN GENERAL

Some actions do not have determinate effects. Asking your boss for a promotion may get you promoted, but it may also get you fired, so this action can be pictured like this:

- Employed
- Asking for promotion
- Promoted
- Fired

Another example: opening a window. This brings about a change in the world, as follows.

- Open window

The action of window-opening changes a state in which the window is closed into one in which it is open. This is more subtle than toggling an alarm clock, for once the window is open a different action is needed to close it again. Also, the action of opening a window can only be applied to closed windows, not to open ones. We say: performing the action has a precondition or presupposition.

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Examples

Some actions do not have determinate effects. Asking your boss for a promotion may get you promoted, but it may also get you fired, so this action can be pictured like this:

employed
promoted
fired
ask for promotion

Another example: opening a window. This brings about a change in the world, as follows.

open window

The action of window-opening changes a state in which the window is closed into one in which it is open. This is more subtle than toggling an alarm clock, for once the window is open a different action is needed to close it again. Also, the action of opening a window can only be applied to closed windows, not to open ones. We say: performing the action has a precondition or presupposition.

In fact, the public announcements from the previous chapter can also be viewed as (communicative) actions covered by our general framework. A public announcement is an action that effects a change in an information model.

\[
\begin{array}{c}
p \\
!p \\
\end{array}
\]
Some actions can be undone by reversing them: the reverse of opening a window is closing it. Other actions are much harder to undo: if you smash a piece of china then it is sometimes hard to mend it again. So here we have a choice: do we assume that basic actions can be undone? If we do, we need an operation for this, for taking the converse of an action. If, in some context, we assume that undoing an action is generally impossible we should omit the converse operation in that context.

Exercise 6.1
Suppose \( \hat{\cdot} \) is used for reversing basic actions. So \( a \hat{\cdot} \) is the converse of action \( a \), and \( b \hat{\cdot} \) is the converse of action \( b \). Let \( a; b \) be the sequential composition of \( a \) and \( b \), i.e., the action that consists of first doing \( a \) and then doing \( b \). What is the converse of \( a; b \)?

6.3 Viewing Actions as Relations
As an exercise in abstraction, we will now view actions as binary relations on a set \( S \) of states. The intuition behind this is as follows. Suppose we are in some state \( s \) in \( S \). Then performing some action \( a \) will result in a new state that is a member of some set of new states \( \{s_1, \ldots, s_n\} \).

If this set is empty, this means that the action \( a \) has aborted in state \( s \). If the set has a single element \( s_0 \), this means that the action \( a \) is deterministic on state \( s \), and if the set has two or more elements, this means that action \( a \) is non-deterministic on state \( s \). The general picture is:

\[
\begin{array}{cccc}
  & s_1 & s_2 & s_3 \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  s & \ldots & \ldots & \ldots & s_n \\
\end{array}
\]

Clearly, when we extend this picture to the whole set \( S \), what emerges is a binary relation on \( S \), with an arrow from \( s \) to \( s_0 \) (or equivalently, a pair \((s, s_0)\) in the relation) just in case performing action \( a \) in state \( s \) may have \( s_0 \) as result. Thus, we can view binary relations on \( S \) as the interpretations of basic action symbols \( a \).

The set of all pairs taken from \( S \) is called \( S \times S \), or \( S^2 \). A binary relation on \( S \) is simply a set of pairs taken from \( S \), i.e., a subset of \( S^2 \). Given this abstract interpretation of basic relations, it makes sense to ask what corresponds to the operations on actions that we encountered in Section 6.2. Let's consider them in turn.
Semantics for Propositional Modal Logic

1. Relational semantics (i.e., Kripke semantics)
2. Algebraic semantics (BAO: Boolean algebras with operators)
3. Topological semantics (Closure algebras)
4. Category-theoretic (Coalgebras)
Semantics for Propositional Modal Logic

1. Relational semantics (i.e., Kripke semantics)
2. Algebraic semantics (BAO: Boolean algebras with operators)
3. Topological semantics (Closure algebras)
4. Category-theoretic (Coalgebras)
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false?
   - true.

2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false?
   - false.

3. Is \(A \rightarrow (B \lor C)\) true or false?
   - It depends!

4. Is \(\Box A \rightarrow (B \rightarrow \Box A)\) true or false?
   - true.

5. Is \(\neg \Box A \land \neg (\Diamond B \lor \neg \Box A)\) true or false?
   - false.

6. Is \(\neg \Box A \land \neg (\Diamond B \lor \Diamond \neg A)\) true or false?
   - false.
   - (tricky: \(\neg \Diamond \neg A\) is equivalent to \(\Box A\).)

7. Is \(\Box A \rightarrow A\) true or false?
   - It depends!
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false?
Some Warm-up Questions

1. Is $A \rightarrow B \lor B \rightarrow A$ true or false? true.
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? true.

2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false?
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? true.

2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false? false.
Some Warm-up Questions

1. Is \((A \to B) \lor (B \to A)\) true or false? true.

2. Is \(A \to (B \to \neg A)\) true or false? false.

3. Is \(A \to (B \lor C)\) true or false?

(tricky: \(\neg \Diamond \neg A\) is equivalent to \(\Box A\).)
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? true.
2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false? false.
3. Is \(A \rightarrow (B \lor C)\) true or false? It depends!
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? true.
2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false? false.
3. Is \(A \rightarrow (B \lor C)\) true or false? It depends!
4. Is \(\square A \rightarrow (B \rightarrow \square A)\) true or false?
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? true.

2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false? false.

3. Is \(A \rightarrow (B \lor C)\) true or false? It depends!

4. Is \(\Box A \rightarrow (B \rightarrow \Box A)\) true or false? true.
Some Warm-up Questions

1. Is \((A \to B) \lor (B \to A)\) true or false? true.

2. Is \(A \to (B \to \neg A)\) true or false? false.

3. Is \(A \to (B \lor C)\) true or false? It depends!

4. Is \(\square A \to (B \to \square A)\) true or false? true.

5. Is \(\neg \square A \land \neg (\diamond B \lor \neg \square A)\) true or false?
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? true.

2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false? false.

3. Is \(A \rightarrow (B \lor C)\) true or false? It depends!

4. Is \(\Box A \rightarrow (B \rightarrow \Box A)\) true or false? true.

5. Is \(\neg \Box A \land \neg (\Diamond B \lor \neg \Box A)\) true or false? false.
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? true.

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5. Is \(\neg \Box A \land \neg (\Diamond B \lor \neg \Box A)\) true or false? false.

6. Is \(\neg \Box A \land \neg (\Diamond B \lor \Diamond \neg A)\) true or false?
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? *true.*

2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false? *false.*

3. Is \(A \rightarrow (B \lor C)\) true or false? *It depends!*

4. Is \(\Box A \rightarrow (B \rightarrow \Box A)\) true or false? *true.*

5. Is \(\neg \Box A \land \neg (\Diamond B \lor \neg \Box A)\) true or false? *false.*

6. Is \(\neg \Box A \land \neg (\Diamond B \lor \Diamond \neg A)\) true or false? *false.*
   *(tricky: \(\neg \Diamond \neg A\) is equivalent to \(\Box A\).)*
Some Warm-up Questions

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false? true.

2. Is \(A \rightarrow (B \rightarrow \neg A)\) true or false? false.

3. Is \(A \rightarrow (B \lor C)\) true or false? It depends!

4. Is \(\Box A \rightarrow (B \rightarrow \Box A)\) true or false? true.

5. Is \(\neg \Box A \land \neg (\Diamond B \lor \neg \Box A)\) true or false? false.

6. Is \(\neg \Box A \land \neg (\Diamond B \lor \Diamond \neg A)\) true or false? false.
   (tricky: \(\neg \Diamond \neg A\) is equivalent to \(\Box A\).)

7. Is \(\Box A \rightarrow A\) true or false?
Some Warm-up Questions

1. Is \( (A \rightarrow B) \lor (B \rightarrow A) \) \text{true} or \text{false}? \text{true.}

2. Is \( A \rightarrow (B \rightarrow \neg A) \) \text{true} or \text{false}? \text{false.}

3. Is \( A \rightarrow (B \lor C) \) \text{true} or \text{false}? \text{It depends!}

4. Is \( \Box A \rightarrow (B \rightarrow \Box A) \) \text{true} or \text{false}? \text{true.}

5. Is \( \neg \Box A \land \neg (\Diamond B \lor \neg \Box A) \) \text{true} or \text{false}? \text{false.}

6. Is \( \neg \Box A \land \neg (\Diamond B \lor \Diamond \neg A) \) \text{true} or \text{false}? \text{false. (tricky: \( \neg \Box A \) is equivalent to \( \Box A \).)}

7. Is \( \Box A \rightarrow A \) \text{true} or \text{false}? \text{It depends!}
Kripke Structures

The main idea:

▶ 'It is sunny outside' is currently true, but it is not necessary (for example, if we were currently in Amsterdam).

▶ We say $\phi$ is necessary provided $\phi$ is true in all (relevant) situations (states, worlds, possibilities).

▶ A Kripke structure is

1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
2. A relation on the set of states (specifying the "relevant situations")
Kripke Structures

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▸ ‘It is sunny outside’ is currently true
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- A Kripke structure is
  1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
  2. A relation on the set of states (specifying the “relevant situations”)
A Kripke Structure

1. Set of states

- $w_1$
- $w_2$
- $w_3$
- $w_4$
- $w_5$
A Kripke Structure

1. Set of states (propositional valuations)
A Kripke Structure

1. Set of states (propositional valuations)
2. Accessibility relation
A Kripke Structure

1. Set of states (propositional valuations)
2. Accessibility relation

- Set of states:
  - $w_1$: $A$
  - $w_2$: $B$
  - $w_3$: $B$
  - $w_4$: $B, C$
  - $w_5$: $A, B$

- Accessibility relation:
  - $w_3 R w_5$
Truth of Modal Formulas

**Model:** $\mathcal{M} = \langle W, R, V \rangle$ where $W \neq \emptyset$, $R \subseteq W \times W$ and $V : \text{At} \rightarrow \wp(W)$ (At is the set of atomic propositions).
Truth of Modal Formulas

Model: \( \mathcal{M} = \langle W, R, V \rangle \) where \( W \neq \emptyset \), \( R \subseteq W \times W \) and \( V : \text{At} \rightarrow \wp(W) \) (At is the set of atomic propositions).

Truth at a state in a model: \( \mathcal{M}, w \models \varphi \)
- \( \mathcal{M}, w \models p \) iff \( w \in V(p) \)
- \( \mathcal{M}, w \models \neg \varphi \) iff \( \mathcal{M}, w \not\models \varphi \)
- \( \mathcal{M}, w \models \varphi \land \psi \) iff \( \mathcal{M}, w \models \varphi \) and \( \mathcal{M}, w \models \psi \)
- \( \mathcal{M}, w \models \Box \varphi \) iff for all \( v \in W \), if \( wRv \) then \( \mathcal{M}, v \models \varphi \)
- \( \mathcal{M}, w \models \Diamond \varphi \) iff there is a \( v \in W \) such that \( \mathcal{M}, v \models \varphi \)
Example

\[
\begin{align*}
&\text{Modal Logic} \\
\end{align*}
\]
Example

Modal Logic
Example

$w_3 \models \Box B$

Diagram:

- $w_1$ from $A$ to $B$
- $w_2$ from $B$ to $B$
- $w_3$ from $B$ to $B$
- $w_4$ from $B$ to $B, C$
- $w_5$ from $A, B$ to $A, B$

Nodes: $A$, $B$, $B, C$, $A, B$
Example

\[ w_3 \models \Box B \]
Example

\[ w_3 \models \Box C \]
Example

\[ w_1 \quad A \quad w_2 \quad B \quad w_3 \quad B \quad w_4 \quad B, C \quad w_5 \quad A, B \]

\( w_3 \vdash \Diamond C \)
Example

\[ w_3 \not\models \square C \]
Example

\[ w_3 \not\models \Box C \]
Example

$w_1 \models \Diamond \Box B$
Example

$w_1 \models \lozenge \Box B$
Example

$w_1 \models \diamondbox \square B$
Example

\[ w_5 \models \Box C \]
Example

\( w_5 \models \Box (B \land \neg B) \)
Example

$w_1 \rightarrow A \rightarrow B \rightarrow B, C \rightarrow B \rightarrow A, B$ with $w_5 \models \neg \Diamond B$.
\[
\begin{align*}
\text{w}_1 & = \square B \land B? \\
\text{w}_1 & = \lozenge \lozenge B? \\
\text{w}_1 & = \lozenge \lozenge \lozenge B? \\
\text{w}_1 & = \square \square B? \\
\text{w}_1 & = \square \lozenge C? \\
\text{w}_1 & = \lozenge \lozenge C?
\end{align*}
\]
\( w_1 \not\models \Box B \land B \)
\( w_1 \models \Diamond \Diamond B? \)
\( w_1 \models \Diamond \Diamond \Diamond B? \)
\( w_1 \models \Box \Box B? \)
\( w_1 \models \Box \Diamond C? \)
\( w_1 \models \Diamond \Diamond C? \)
$w_1 \nmid \square B \land B$

$w_1 \models \lozenge \lozenge B$

$w_1 \models \lozenge \lozenge \lozenge B$

$w_1 \models \square \square B?$

$w_1 \models \square \lozenge C?$

$w_1 \models \lozenge \lozenge C?$

Modal Logic 36/45
Modal Logic 36/45
$w_1 \not\models \Box B \land B$

$w_1 \models \Diamond \Diamond B$

$w_1 \models \Diamond \Diamond \Diamond B$

$w_1 \not\models \Box \Box B$

$w_1 \models \Box \Diamond C$?

$w_1 \models \Diamond \Diamond C$?
\[ w_1 \not\models \Box B \land B \]
\[ w_1 \models \Diamond \Diamond B \]
\[ w_1 \models \Diamond \Diamond \Diamond B \]
\[ w_1 \not\models \Box \Box B \]
\[ w_1 \not\models \Box \Diamond C \]
\[ w_1 \models \Diamond \Diamond C ? \]
$w_1 \not\models \Box B \land B$

$w_1 \models \Diamond \Diamond B$

$w_1 \models \Diamond \Diamond \Diamond B$

$w_1 \models \Box \Box B$

$w_1 \not\models \Box \Diamond C$

$w_1 \models \Diamond \Diamond C$?
\( w_1 \not\models \square B \land B \)
\( w_1 \models \diamond \diamond B \)
\( w_1 \models \diamond \diamond \diamond B \)
\( w_1 \models \square \square B \)
\( w_1 \not\models \square \diamond C \)
\( w_1 \models \diamond \diamond C \)
$w_1 \models \diamond B \land B$

$w_1 \models \diamond \diamond B$

$w_1 \models \diamond \diamond \diamond B$

$w_1 \models \Box \Box B$

$w_1 \not\models \Box \Box C$

$w_1 \models \diamond \diamond C$
\(\Box(A \rightarrow B) \) vs. \(A \rightarrow \Box B\)
\[ (A \rightarrow B) \text{ vs. } A \rightarrow \Box B \]

\begin{itemize}
  \item \( w \models X \rightarrow Y \) provided either \( w \not\models X \) or \( w \models Y \)
\end{itemize}
□(A → B) vs. A → □B

\[ w_1 \models □(A → B) \]

\[ w \models X → Y \text{ provided either } w \not\models X \text{ or } w \models Y \]
$\Box(A \rightarrow B)$ vs. $A \rightarrow \Box B$

$w_1 \models \Box(A \rightarrow B)$

$w \models X \rightarrow Y$ provided either $w \not\models X$ or $w \models Y$
\(\square(A \rightarrow B)\) vs. \(A \rightarrow \square B\)

\[w_1 \models \square(A \rightarrow B)\ and\ w_1 \not\models A \rightarrow \square B\]

\[w \models X \rightarrow Y\ provided\ either\ w \not\models X\ or\ w \models Y\]
Some Facts

- □φ ∨ ¬□φ is always true (i.e., true at any state in any Kripke structure), but what about □φ ∨ □¬φ?
Some Facts

- $\square \varphi \lor \neg \square \varphi$ is always true (i.e., true at any state in any Kripke structure), but what about $\square \varphi \lor \square \neg \varphi$?

- $\square \varphi \land \square \psi \rightarrow \square (\varphi \land \psi)$ is true at any state in any Kripke structure.
Some Facts

- □φ ∨ ¬□φ is always true (i.e., true at any state in any Kripke structure), but what about □φ ∨ □¬φ?

- □φ ∧ □ψ → □(φ ∧ ψ) is true at any state in any Kripke structure. What about □(φ ∨ ψ) → □φ ∨ □ψ?
Some Facts

- □φ ∨ ¬□φ is always true (i.e., true at any state in any Kripke structure), but what about □φ∨□¬φ?

- □φ ∧ □ψ → □(φ ∧ ψ) is true at any state in any Kripke structure. What about □(φ ∨ ψ) → □φ ∨ □ψ?

- □φ ↔ ¬◊¬φ is true at any state in any Kripke structure.
More Facts

Determine which of the following formulas are always true at any state in any Kripke structure:

1. $\square \varphi \rightarrow \Diamond \varphi$
2. $\square (\varphi \lor \neg \varphi)$
3. $\square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)$
4. $\square \varphi \rightarrow \varphi$
5. $P \rightarrow \square \Diamond \varphi$
6. $\Diamond (\varphi \lor \psi) \rightarrow \Diamond \varphi \lor \Diamond \psi$
But, we are not always interested in all Kripke structures.

Eg., for each state $w$, $w$ is accessible from itself ($R$ is a reflexive relation).
But, we are not always interested in all Kripke structures.

For example, consider the epistemic interpretation: A state $v$ is accessible from $w$ ($wRv$) provided “given the agents information, $w$ and $v$ are indistinguishable”.
But, we are not always interested in all Kripke structures.

For example, consider the epistemic interpretation: A state $v$ is accessible from $w$ ($wRv$) provided “given the agents information, $w$ and $v$ are indistinguishable”. What are natural properties?
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For example, consider the epistemic interpretation: A state $v$ is accessible from $w$ ($wRv$) provided “given the agents information, $w$ and $v$ are indistinguishable”. What are natural properties?

Eg., for each state $w$, $w$ is accessible from itself ($R$ is a reflexive relation).
But, we are not always interested in all Kripke structures.

For example, consider the epistemic interpretation: A state $v$ is accessible from $w$ ($wRv$) provided “given the agents information, $w$ and $v$ are indistinguishable”. What are natural properties?

Eg., for each state $w$, $w$ is accessible from itself ($R$ is a reflexive relation).

Some Facts

- $\Box \varphi \rightarrow \varphi$ is true at any state in any Kripke structure where each state is accessible from itself.
- $\Box \varphi \rightarrow \Diamond \varphi$ is true at any state in any Kripke structure where each state has at least one accessible world.
Can you think of properties that force each of the following formulas to be true at any state in any appropriate Kripke structure?

1. $\diamond \varphi \rightarrow \square \varphi$
2. $\square \varphi \rightarrow \square \square \varphi$
Defining States

\[ w_1 \]
\[ w_2 \]
\[ w_3 \]
\[ w_4 \]

- \( w_4 \models \)
- \( w_3 \models \)
- \( w_2 \models \)
- \( w_1 \models \)
Defining States

\[ w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow w_4 \]

- \( w_4 \models \Box \bot \)
- \( w_3 \models \)
- \( w_2 \models \)
- \( w_1 \models \)
Defining States

\[
\begin{align*}
& \models w_4 \vdash \Box \bot \\
& \models w_3 \vdash \Diamond \Box \bot \land \Box \Box \bot \\
& \models w_2 \vdash \\
& \models w_1 \vdash
\end{align*}
\]
Defining States

- $w_4 \models \Box \bot$
- $w_3 \models \Diamond \Box \bot \land \Box \Box \bot$
- $w_2 \models \Diamond \Box \bot \land \Diamond \Diamond \top$
- $w_1 \models \bot$
Defining States

- $w_4 \models \Box \bot$
- $w_3 \models \Diamond \Box \bot \land \Box \Box \bot$
- $w_2 \models \Diamond \Box \bot \land \Diamond \Diamond \top$
- $w_1 \models \Diamond (\Diamond \Box \bot \land \Box \Box \bot)$
Defining States

\[
\begin{align*}
\mathcal{W}_4 & \models \Box \bot \\
\mathcal{W}_3 & \models (\Diamond \bot \wedge \Box \bot) \\
\mathcal{W}_2 & \models (\Diamond \bot \wedge \Box \bot) \\
\mathcal{W}_1 & \models (\Diamond \bot \wedge \Box \bot) 
\end{align*}
\]
Defining States

\[ w_1 \]

\[ w_2 \]

\[ w_3 \]

\[ w_4 \]

- \( w_4 \models □⊥ \)
- \( w_3 \models ◊□⊥ ∧ □□⊥ \)
- \( w_2 \models ◊□⊥ ∧ ◊◊\top \)
- \( w_1 \models ◊(◊□⊥ ∧ □□⊥) \)
What is the difference between states \( w_1 \) and \( v_1 \)?
Distinguishing States

What is the difference between states $w_1$ and $v_1$?
Is there a modal formula true at $w_1$ but not at $v_1$?
Distinguishing States

\[ w_1 \models \Box \Diamond \neg A \] but \[ v_1 \not\models \Box \Diamond \neg A. \]
Distinguishing States

\[ w_1 \models \square \Diamond \neg A \text{ but } v_1 \not\models \square \Diamond \neg A. \]
Distinguishing States

$$w_1 \models \Box \Diamond \neg A$$ but $$v_1 \not\models \Box \Diamond \neg A.$$
Distinguishing States

$w_1 \models \square \Diamond \neg A$ but $v_1 \not\models \square \Diamond \neg A$. 
Distinguishing States

\[ w_1 \models \Box \Diamond \neg A \text{ but } v_1 \not\models \Box \Diamond \neg A. \]
What about now? Is there a modal formula true at $w_1$ but not $v_1$?
Distinguishing States

No modal formula can distinguish $w_1$ and $v_1$!
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?

\[ \square (\square \bot \lor \Diamond \square \bot) \]

- **K**: States \( s \) and \( t \)
- **M**: States \( t \) and \( u \)
- **N**
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?

□(□⊥ ∨ ◊□⊥)
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?

\[ \Box (\Box \perp \lor \Diamond \Box \perp) \]
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?

\[ \square \left( \square \bot \lor \lozenge \square \bot \right) \]
Which pair of states cannot be distinguished by a modal formula?
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?

\[ \Box(\Box \bot \lor \Diamond \Box \bot) \]
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?

$$\Box (\Box \bot \lor \Diamond \Box \bot)$$
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?

\[ \square\left(\square \bot \lor \diamond \square \bot\right) \]
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?

\[ \Box (\Box \bot \lor \Diamond \Box \bot) \]

K  M  N

s  K  t  M  u
Next time: Chapters 3 & 4.

Questions?
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