1 Ultrafilter Extensions

Definition 1.1 (Ultrafilter) Let \( W \) be a non-empty set. An ultrafilter on \( W \) is a set \( u \subseteq \mathcal{P}(W) \) satisfying the following properties:

1. \( \emptyset \not\in u \)
2. If \( X, Y \in u \) then \( X \cap Y \in u \)
3. If \( X \in u \) and \( X \subseteq Y \) then \( Y \in u \).
4. For all \( X \subseteq W \), either \( X \in u \) or \( X \not\in u \) (where \( \overline{X} \) is the complement of \( X \) in \( W \)).

A collection \( u_0 \subseteq \mathcal{P}(W) \) has the **finite intersection property** provided for each \( X, Y \in u_0 \), \( X \cap Y \neq \emptyset \).

Theorem 1.2 (Ultrafilter Theorem) If a set \( u_0 \subseteq \mathcal{P}(W) \) has the finite intersection property, then \( u_0 \) can be extended to an ultrafilter over \( W \) (i.e., there is an ultrafilter \( u \) over \( W \) such that \( u_0 \subseteq u \)).

**Proof.** Suppose that \( u_0 \) has the finite intersection property. Then, consider the set

\[ u_1 = \{ Z \mid \text{there are finitely many sets } X_1, \ldots, X_k \text{ such that } Z = X_1 \cap \cdots \cap X_k \} . \]

That is, \( u_1 \) is the set of finite intersections of sets from \( u_0 \). Note that \( u_0 \subseteq u_1 \), since \( u_0 \) has the finite intersection property, we have \( \emptyset \not\in u_1 \), and by definition \( u_1 \) is closed under finite intersections. Now, define \( u_2 \) as follows:

\[ u' = \{ Y \mid \text{there is a } Z \in u_1 \text{ such that } Z \subseteq Y \} . \]
We claim that $u'$ is a consistent filter: $Y_1, Y_2 \in u'$ then there is a $Z_1 \in u_1$ such that $Z_1 \subseteq Y_1$ and $Z_2 \in u_1$ such that $Z_2 \subseteq Y_2$. Then, since $\emptyset \neq Z_1 \cap Z_2 \in u_1$, we have $Z_1 \cap Z_2 \subseteq Y_1 \cap Y_2$. Hence, $Y_1 \cap Y_2 \in u_1$. Also, if $X \in u_1$ then there is a $Z \in u_1$ such that $Z \subseteq X$. If $X \subseteq Y$, then $Z \subseteq Y$ and so $Y \in u_1$. Hence, $u_1$ is a consistent filter.

The next step is to show that $u_1$ can be extended to an ultrafilter. This follows almost directly from Zorn’s Lemma\(^1\): Consider the set $Z$ of all filters that extend $u_1$. That is, $Z = \{ v \mid u_1 \subseteq v \text{ and } v \text{ is a consistent filter} \}$. Note that $Z$ is partially-ordered under the $\subseteq$-relation. Furthermore, the upper bound of any chain in $Z$ (i.e., sequence of ultrafilters $v_0 \subseteq v_1 \subseteq \cdots$) is the union of all the filters in the chain. This collection of sets will be a consistent ultrafilter extending $u_1$, and so is contained in $Z$. By Zorn’s Lemma, $Z$ must contain a maximal element. This maximal element must be an ultrafilter (containing $u_1$).

\[ \text{QED} \]

Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. Two functions are relevant to our analysis:

- $m : \wp(W) \to \wp(W)$ defined as $m(X) = \{ w \mid \text{there is a } v \text{ such that } wRv \text{ and } v \in X \}$, and
- $l : \wp(W) \to \wp(W)$ defined as $l(X) = \{ w \mid \text{for all } v, \text{ if } wRv \text{ then } v \in X \}$.

**Definition 1.3 (Ultrafilter Extension)** An ultrafilter extension is a model $ue(\mathcal{M}) = \langle Uf(W), R^{ue}, V^{ue} \rangle$ where

- $Uf(W) = \{ u \mid u \text{ is an ultrafilter over } W \}$,
- $uR^{ue}u'$ iff for all $X \subseteq W$, if $X \in u'$ then $m(X) \in u$, and
- $V^{ue}(p) = \{ u \mid V(p) \in u \}$.

**Fact 1.4** In an ultrafilter extension $ue(\mathcal{M}) = \langle Uf(W), R^{ue}, V^{ue} \rangle$, we have $uR^{ue}u'$ iff $\{ Y \mid l(Y) \in u \} \subseteq u'$.

**Proof.** Left as an exercise. \text{QED}

Let $\mathcal{M} = \langle W, R, V \rangle$ be a model. The truth map $[\cdot]_M : \mathcal{L} \to \wp(W)$ is defined by induction on the structure of $\varphi$ as follows:

- For $p \in \text{At}$, $[p]_M = V(p)$

---

\(^1\)Zorn’s Lemma states that in a partially ordered set $P$, if every chain has an upper bound in $P$, then $P$ contains at least one maximal element. The proof of Zorn’s Lemma uses the Axiom of Choice (indeed, it is equivalent to the Axiom of Choice).
• $\lnot \varphi_M = W - \varphi_M$
• $[\varphi \land \psi]_M = [\varphi]_M \land [\psi]_M$
• $[\Diamond \varphi]_M = m([\varphi]_M)$

**Lemma 1.5** For all models $M = \langle W, R, V \rangle$, for all modal formulas $\varphi$, we have $[\varphi]_M \in u$ iff $ue(M), u \models \varphi$.

**Proof.** The proof is by induction on the structure of $\varphi$. Then we have,

**Base case:** $\varphi$ is $p \in \text{At}$.

$$[p]_M \in u \quad \text{iff} \quad V(p) \in u \quad \text{(Definition of} \; [\cdot]_M)$$

$$\text{iff} \quad u \in V^{ue}(p) \quad \text{(Definition of} \; V^{ue})$$

$$\text{iff} \quad ue(M), u \models p \quad \text{(Definition of truth in a model)}$$

**Induction Hypothesis:** For all $\psi$ less complex than $\varphi$, $[\psi]_M \in u$ iff $ue(M), u \models \psi$.

**Case 1:** $\varphi$ is $\lnot \psi$:

$$[\lnot \psi]_M \in u \quad \text{iff} \quad W - \psi_M \in u \quad \text{(Definition of} \; [\cdot]_M)$$

$$\text{iff} \quad [\psi]_M \not\in u \quad \text{(Properties of an ultrafilter)}$$

$$\text{iff} \quad ue(M), u \not\models \psi \quad \text{(Induction Hypothesis)}$$

$$\text{iff} \quad ue(M), u \models \lnot \psi \quad \text{(Definition of truth)}$$

**Case 2:** $\varphi$ is $\psi_1 \land \psi_2$

$$[\psi_1 \land \psi_2]_M \in u \quad \text{iff} \quad [\psi_1]_M \cap [\psi_2]_M \in u \quad \text{(Definition of} \; [\cdot]_M)$$

$$\text{iff} \quad [\psi_1]_M \in u \text{ and } [\psi_2]_M \in u \quad \text{(Properties of ultrafilters)}$$

$$\text{iff} \quad ue(M), u \models \psi_1 \text{ and } ue(M), u \models \psi_2 \quad \text{(Induction hypothesis)}$$

$$\text{iff} \quad ue(M), u \models \psi_1 \land \psi_2 \quad \text{(Definition of truth)}$$

**Case 3:** $\varphi$ is $\Diamond \psi$

Suppose that $ue(M), u \models \Diamond \psi$. Then, there is a $u' \in Uf(W)$ such that $uR^{ue}u'$ and $ue(M), u' \models \psi$. By the induction hypothesis, $[\psi]_M \in u'$. By the definition of $R^{ue}$, we have $m([\psi]_M) \in u$, and so, $[\Diamond \psi]_M \in u$. Thus, we have shown that $ue(M), u \models \Diamond \psi$ implies $[\Diamond \psi]_M \in u$.

Suppose that $[\Diamond \psi]_M \in u$. We must show $ue(M), u \models \Diamond \psi$. Consider the set

$$u_0 = \{Y \mid l(Y) \in u \} \cup \{\psi\}_M$$

---

2Less complex means that $\psi$ contains fewer connectives.
We claim that \( u_0 \) has the finite intersection property. We first show that \( \{ Y \mid l(Y) \in u \} \) is closed under finite intersections. It is enough to show that for any two sets \( Y_1, Y_2 \in \{ Y \mid l(Y) \in u \} \), \( Y_1 \cap Y_2 \in \{ Y \mid l(Y) \in u \} \) (why?). Suppose that \( Y_1, Y_2 \in \{ Y \mid l(Y) \in u \} \). Note that \( l(Y_1 \cap Y_2) = l(Y_1) \cap l(Y_2) \). Then, since \( u \) is an ultrafilter and \( l(Y_1), l(Y_2) \in u \), we have \( l(Y_1 \cap Y_2) = l(Y_1) \cap l(Y_2) \in u \). Hence \( Y_1 \cap Y_2 \in \{ Y \mid l(Y) \in u \} \). Next we show that for any \( Z \in \{ Y \mid l(Y) \in u \} \), we have \( Z \cap \{ \psi \}_{M} \neq \emptyset \). Choose an arbitrary \( Z \) such that \( l(Z) \in u \). We will show \( Z \cap \{ \psi \}_{M} \neq \emptyset \). Since \( l(Z) \in u \) and \( \{ \psi \}_{M} \subseteq u \), we have \( l(Z) \cap \{ \psi \}_{M} \in u \), and so \( l(Z) \cap \{ \psi \}_{M} \neq \emptyset \). Let \( w \in l(Z) \cap \{ \psi \}_{M} \). Then, there is a \( v \in W \) such that \( M, v \models \psi \). I.e., \( wRv \) and \( v \in \{ \psi \}_{M} \). Since \( w \in l(Y) \) and \( wRv \), we have \( v \in Y \cdot \) Hence, \( v \in Y \cap \{ \psi \}_{M} \). This implies that \( u_0 \) has the finite intersection property (why?). By the ultrafilter theorem, there is an ultrafilter \( u' \) such that \( u_0 \subseteq u' \). Since \( \{ Y \mid l(Y) \in u \} \subseteq u' \), we have \( u_{R}^{ue}u' \). By the induction hypothesis, since \( \{ \psi \}_{M} \subseteq u' \), we have \( u_{w}(M), u' \models \psi \). Hence, \( u_{w}(M), u \models \Diamond \psi \). QED

**Corollary 1.6** For all models \( M \) and states \( w \in M \), we have \( w \hookrightarrow u_{w} \), where \( u_{w} \) is the principle ultrafilter generated by \( w \).

**Proof.** Let \( M = (W, R, V) \) be a model and \( w \in W \). The principle ultrafilter generated by \( w \) is \( u_{w} = \{ X \subseteq W \mid w \in X \} \). Let \( \varphi \) be an arbitrary modal formula. We have \( M, w \models \varphi \) iff \( w \in [\varphi]_{M} \) iff \( [\varphi]_{M} \in u_{w} \) iff \( u_{w}(M), u_{w} \models \varphi \) (the latter equivalence follows from the above Lemma). QED

**Lemma 1.7** For all models \( M \), \( u_{w}(M) \) is \( m \)-saturated.

**Proof.** Suppose that \( u_{w}(M) = (Uf(W), R^{ue}, V^{ue}) \) an ultrafilter extension of some model \( M = (W, R, V) \). Let \( u \in Uf(W) \) be any state in \( u_{w}(M) \) and \( \Sigma \) be a arbitrary set of modal formulas. Suppose that every finite subset of \( \Sigma \) is satisfiable at some successor of \( u \) (i.e., for each finite set \( \Delta \subseteq \Sigma \), there is a state \( v_{\Delta} \in W \) such that \( uR^{ue}v_{\Delta} \) and \( u_{w}(M), v_{\Delta} \models \Delta \)). We must find a state \( v \in W \) such that \( uR^{ue}v \) and \( u_{w}(M), v \models \Sigma \) (i.e., for each \( \psi \in \Sigma \), \( u_{w}(M), v \models \psi \)). Consider the set

\[
v_{0} = \{ Y \mid l(Y) \in u \} \cup \{ [\psi]_{M} \mid \psi \in \Sigma \}
\]

We will show \( v_{0} \) has the finite intersection property. Since \( \{ Y \mid l(Y) \in u \} \) is closed under finite intersections, it is enough to show that \( Y \cap \{ [\psi]_{M} \mid \psi \in \Delta \} \neq \emptyset \) for some finite subset \( \Delta \) of \( \Sigma \). Note that \( \bigcap \{ [\psi]_{M} \mid \psi \in \Delta \} = [\bigwedge \Delta]_{M} \). Recall that \( \Delta \) is satisfiable at some successor state \( v_{\Delta} \) of \( u \). That is, \( uR^{ue}v_{\Delta} \) and \( u_{w}(M), v_{\Delta} \models \Delta \). By Lemma 1.5, this means \( [\bigwedge \Delta]_{M} \in v_{\Delta} \). By the definition of \( R^{ue} \), we have \( m([\bigwedge \Delta]_{M}) \in u \). Hence, \( [\bigwedge \Delta]_{M} \in u \). Since \( u \) is
an ultrafilter, we have \( l(Y) \cap \Diamond \Delta \subseteq u \). Hence, (since \( \emptyset \notin u \)) there is a \( x \in l(Y) \cap \Diamond \Delta \). This implies there is a \( y \) such that \( xRy \) and \( y \in \Diamond \Delta \). Since \( xRy \) and \( x \in l(Y) \), we have \( y \in Y \). This means that \( y \in Y \cap \Diamond \Delta \). Hence, \( v_0 \) has the finite intersection property.

By the ultrafilter theorem there is an ultrafilter \( v \) such that \( v_0 \subseteq v \). By construction \( v \) is a successor \( u \) (i.e., \( uRv \)) and by Lemma 1.5, we have for each \( \psi \in \Sigma \), \( ue(M), u \models \psi \). Hence, \( \Sigma \) is satisfiable is some successor state of \( u \). QED

**Theorem 1.8 (Bisimulation Somewhere Else Theorem)** For all models \( M \) and \( M' \), we have \( M, w \leftrightarrow M', w' \) iff \( ue(M), u_w \leftrightarrow ue(M'), u_{w'} \), where \( u_w \) and \( u_{w'} \) are the principle ultrafilters containing \( w \) and \( w' \) respectively.

**Proof.** Suppose that \( ue(M), u_w \leftrightarrow ue(M'), u_{w'} \). Let \( \varphi \) be any model formula. We have

\[
M, w \models \varphi \quad \text{iff} \quad ue(M), u_w \models \varphi \quad \text{(Corollary 1.6)}
\]

\[
\text{iff} \quad ue(M'), u_{w'} \models \varphi \quad \text{(Bisimulation implies modal equivalence)}
\]

\[
\text{iff} \quad M', w' \models \varphi \quad \text{(Corollary 1.6)}
\]

Suppose that \( M, w \leftrightarrow M', w' \). Then, by Corollary 1.6, we have \( ue(M), u_w \leftrightarrow ue(M'), u_{w'} \). By Lemma 1.7, both \( ue(M) \) and \( ue(M') \) are modally saturated. In modally saturated models, modal equivalence implies bisimilarity. Hence, \( ue(M), u_w \leftrightarrow ue(M'), u_{w'} \). QED

### 2 The Standard Translation

Let \( M = \langle W,R,V \rangle \) be a Kripke model. The first-order language \( L_1 \) is built from a signature containing unary predicate symbols \( Px \) corresponding to each \( p \in At \) and a binary predicate symbol \( Rxy \). The standard translation is defined as follows:

**Definition 2.1 (Standard Translation)** The standard translation are functions \( st_x : L \rightarrow L_1 \) defined as follows:

\[
\begin{align*}
st_x(p) &= Px \\
st_x(\neg \varphi) &= \neg st_x(\varphi) \\
st_x(\varphi \land \psi) &= st_x(\varphi) \land st_x(\psi) \\
st_x(\Box \varphi) &= \forall y (Rxy \rightarrow st_y(\varphi)) \\
st_x(\Diamond \varphi) &= \exists y (Rxy \land st_y(\varphi))
\end{align*}
\]
Observation 2.2  

Modal logic falls in the two-variable fragment of $L_1$.

Proof. By carefully reusing bound variables, once can ensure that the translation of a modal formula uses only two variable. An example suffices to show how this works:

$$st_x(\Box p) = \exists y (Rxy \land st_y(\Box p)) = \exists y (Rxy \land (\exists x (Ryx \land Px)))$$

QED

Lemma 2.3  Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. For each $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M} \models st_x(\varphi)[x/w]$, where $\models$ denotes truth of $L_1$ in a model $\mathcal{M}$ (viewed as a first-order structure).

Proof. The simple but instructive proof is left to the reader. QED

Theorem 2.4 (Van Benthem Characterization Theorem)  A first-order formula $\alpha(x)$ (in the appropriate language) is invariant for bisimulation iff it is equivalent to the translation of a modal formula.