Bidding Clubs in First-Price Auctions*

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Abstract

We introduce a class of mechanisms, called *bidding clubs*, that allow agents to coordinate their bidding in auctions. Bidding clubs invite a set of agents to join, and each invited agent freely chooses whether to accept the invitation or to participate independently in the auction. Agents who join a bidding club first conduct a "knockout auction" within the club; depending on the outcome of the knockout auction some subset of the members of the club bid in the primary auction in a prescribed way. We model this setting as a Bayesian game, including agents' choices of whether or not to accept a bidding club's invitation. After describing this general setting, we examine the specific case of bidding clubs for first-price auctions. We show the existence of a Bayes-Nash equilibrium where agents choose to participate in bidding clubs when invited and truthfully declare their valuations to the coordinator. Furthermore, we show that the existence of bidding clubs benefits all agents (both inside and outside of bidding clubs).

1 Introduction

Most work on auctions concentrates on the design of auction protocols from the seller's perspective, and in particular on optimal (i.e., revenue maximizing) auction design. In this paper we present a class of systems to assist sets of bidders, *bidding clubs.* The idea is similar to the idea behind "buyer clubs": aggregating the market power of individual bidders. Buyer clubs work when buyers' interests are perfectly aligned; the more buyers join in a purchase the lower the price for everyone. In auctions it is relatively easy for multiple agents to cooperate, hiding behind a single auction participant. Intuitively, these bidders can reduce their payment if they win, by causing others to lower their bids in the case of a first-price auction or by possibly removing the second-highest bidder in the case of a second-price auction. However, the situation in auctions is not as simple as in buyer clubs, because while bidders can gain by sharing information, the competitive nature of auctions means that bidders' interests are not aligned. Thus there is a complex strategic relationship among bidders in a bidding club, and bidding club rules must be designed accordingly.

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1.1 Related Work

Below we discuss the most relevant previous work and its relation to ours, noting the relative scarcity of previous work on bidder-centric mechanisms. This work all comes under the umbrella of self-enforcing collusive protocols for non-repeated auctions. *Collusion* is a negative term reflecting a seller-oriented perspective; since we adopt a more neutral stance towards such bidder activities, we use the term *bidding clubs* rather than the terms *bidding rings* and *cartels* that have been used in the past. However, the technical development is not impacted by such subtle differences in moral attitude.

1.1.1 Collusion in Second-Price Auctions

One of the first formal papers to consider collusion in second-price auctions was written by Graham and Marshall [3]. This paper introduces the knockout procedure: agents announce their bids in a knockout auction; only the highest bidder goes to the auction but this bidder must pay a "ring center" the amount of his gain relative to the case where there was no collusion. The ring center pays each agent in advance; the amount of this payment is calculated so that the ring center will budget-balance *ex-ante*, before knowing the agents' valuations.

Graham and Marshall's work has been extended to deal with variations in the knockout procedure, differential payments, and relations to the Shapley value [4]. The case where only some of the agents are part of the cartel is discussed by Mailath and Zemsky [9]. Ungern and Sternberg [14] discuss collusion in second-price auctions where the designated winner of a cartel is not the agent with the highest valuation. Although not presented in any existing work of which we are aware, it is also easy to extend Graham and Marshall's protocol to an environment where multiple cartels may operate in the same auction alongside independent bidders.

1.1.2 Collusion in First-Price Auctions

There is little formal work on collusion in first-price auctions, the most important exception being a very influential paper by McAfee and McMillan [11]. It is the closest in the literature to our work, and indeed we have borrowed some modelling elements from it. Several sections, including the discussion of enforcement and the argument for independent private values as a model of agents' valuations, are directly applicable to our paper. However, the setting introduced in their work assumes that a fixed number of agents participate in the auction and that all agents are part of a single cartel that coordinates its behavior in the auction. The authors show optimal collusion protocols for "weak" cartels (in which transfers between agents are not permitted: all bidders bid the reserve price, using the auctioneer's tie-breaking rule to randomly select a winner) and for "strong" cartels (the cartel holds a knockout auction, the winner of which bids the reserve price in the main auction while all other bidders sit out; the winner distributes some of his gains to other cartel members through side payments). A small part of the paper deals with the case where in addition to a single cartel there are also additional agents. However, results are shown only for two cases: (1) when non-cartel members bid without taking the existence of a cartel into account and (2) when each agent i has valuation $v_i \in \{0, 1\}$. The authors explain that they do not attempt to deal with general strategic behavior in the case where the cartel consists of only a subset of the agents; furthermore, they do not consider the case where multiple cartels can operate in the same auction. Finally, a brief presentation of "cartel-formation games" is related to our discussion of agents' decision of whether or not to accept an invitation to join a bidding club.

1.1.3 Other Work on Collusion

Less formal discussion of collusion in auctions can be found in a wider variety of papers. For example, a survey paper that discusses mechanisms that are likely to facilitate collusion in auctions, as well as methods for the detection of such schemes, can be found in [6]. A discussion and comparison of the stability of rings associated with classical auctions can be found in [13], concentrating on the case where the valuations of agents in the cartel are honestly reported. Collusion is also discussed in other settings, e.g., aiming to influence purchaser behavior in a repeated procurement setting (see [2]) and in the context of general Bertrand or Cournot competition (see [1]).

Our previously published work anticipates some of the results reported here. Specifically, in [7] we considered bidding clubs under the assumptions that only a single bidding club exists, and that bidders who were not invited to join the club are not aware of the possibility that a bidding club might exist. The current paper is an extension and generalization of that earlier work. An extended abstract of the current paper appeared in AAAI-02 [8].

2 Technical Preliminaries

Our goal is to extend on past work on bidder cooperation in first-price auctions to the standard game-theoretic setting in which *all* agents (both cartel members and non-members) are rational, and act in equilibrium based on true knowledge of the economic environment. We also want to increase realism by allowing for the possibilities that more than one cartel will exist (introducing the new wrinkle that cartel members must reason about the behavior of other cartels) and that some agents will not belong to any cartel. Of course we also want to allow for real-numbered valuations drawn from an interval, as compared to the case studied in [11] where valuations take one of only two discrete values.

2.1 Auction Setting

In this section we give a formal description of the auction setting and introduce notation. An economic environment E consists of a finite set of agents who have non-negative valuations for a good at auction, and a distinguished agent

0—the seller or center. Denote the economic environment described here as E_c. Let $\mathcal{T}$ be the set of possible agent types. The type $\tau_i \in \mathcal{T}$ of agent i is the pair $(v_i, s_i) \in V \times \mathcal{S}$. v_i denotes an agent's valuation: his maximal willingness to pay for the good offered by the center. We assume that v_i represents a purely private valuation for the good, and that v_i is selected independently from the other v_j's of other agents from a known distribution, F, having density function f. By s_i we denote agent i's signal: his private information about the number of agents in the auction. The set of possible signals will be varied throughout the paper; in E_c let $\mathcal{S} = \{\varnothing\}$. Note, however, that the economic environment itself is always common knowledge, and so agents always have some information about the number of agents even when they always receive the null signal.

By $p_n^{\tau_i}$ we denote the probability that agent i assigns to there being exactly n agents in the auction, conditioned on his type τ_i. We denote the whole distribution conditioned on i's type as uppercase P^{τ_i}. The utility function of agent i, $u_i : \mathbb{R} \to \mathbb{R}$ is linear, normalized with $u_i(0) = 0$. The utility of agent i (having valuation v_i) when asked to pay t is $v_i - t$ if i is allocated a good, and it is 0 otherwise. Thus, we assume that there are no externalities in agents' valuations and that agents are risk-neutral. $b_i : \mathcal{T} \to \mathbb{R}$ denotes agent i's strategy, a mapping from i's type τ_i to his declaration in the auction. This may be the null declaration, indicating that i will not participate in the auction.

2.2 Classical first-price auctions

It is instructive to consider the reasons why most previous work in collusion has focused on second-price rather than first-price auctions. Since second-price auctions give rise to dominant strategies, and since colluding agents can gain by having other agents drop out without changing their own bidding behavior, it is possible to study collusion in many settings related to these auctions without performing strategic equilibrium analysis. In particular, agents outside a cartel have no reason to change their strategies if they suspect (or even know) that collusion is taking place. In first-price auctions agents who are not part of the cartel must take into account the likelihood of collusion in deciding what they should bid, since their strategy amounts to predicting the second-highest bid conditional on their bid being highest, and this computation depends on the total number of agents. The settings in [11] are largely designed to overcome this problem: e.g., if all agents belong to the cartel, or if non-cartel agents are assumed to play as though collusion is impossible, the question of how cartel members and non-cartel members reason about each other is avoided.

This suggests that the choice of information structure will make a real difference for the study of collusion in first-price auctions. The most familiar is what we will call the "classical" first-price auction, where the number of participants is part of the economic environment (as in E_c). The equilibrium analysis of classical first-price auctions is quite standard[1]:

[1] When we say that n agents participate in the auction we do not count the distinguished agent 0, who is always present.

Proposition 1 *If valuations are selected independently according to the uniform distribution on* $[0,1]$ *then it is a unique symmetric equilibrium for each agent* i *to follow the strategy:*

$$b(v_i) = \frac{n-1}{n} v_i.$$

Using classical equilibrium analysis (e.g., following Riley and Samuelson [12]) classical first-price auctions can be generalized to an arbitrary continuous distribution F.

Proposition 2 *If valuations are selected from a continuous distribution* F *then it is a unique symmetric equilibrium for each agent* i *to follow the strategy:*

$$b(v_i) = v_i - F(v_i)^{-(n-1)} \int_0^{v_i} F(u)^{n-1} du.$$

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In both cases, observe that the strategy is parameterized by valuation, and also depends on information from the economic environment. It will be notationally useful for us to be able to specify the amount of the equilibrium bid as a function of both v and n:

$$b^e(v_i, n) = v_i - F(v_i)^{-(n-1)} \int_0^{v_i} F(u)^{n-1} du. \tag{1}$$

We are interested in constructing a bidding club protocol: a collusive agreement that requires low bidders to drop out of the main auction if they lose in a knockout auction. It is immediately obvious that such collusion is nonsensical in a classical first-price auction. Since all agents have full knowledge of the economic environment, they all know the true number of agents; as a result, it will not matter to agents outside a cartel if cartel members with low valuations drop out, and so the original equilibrium (based on the true number of agents in the environment) will still hold. This seems more of a problem with our auction model than with collusion in first-price auctions *per se*—in practice bidders do not know how many agents have declined to participate, because they don't actually know the number of agents in the economic environment. The next section considers an economic environment that addresses this issue.

2.3 First-price auctions with stochastic number of bidders

One way of modelling agents' uncertainty about the number of opponents they face is to say that the number of participants is chosen stochastically from a probability distribution, and while the number of participants is not known to the individual agents (not being part of the economic environment) the *distribution* is commonly known [10]. This setting requires that we redefine the economic environment; denote the new economic environment as E_s. Let the set of agents who may participate in the economic environment be $\mathcal{A} \equiv \mathbb{N}$. Let

β_A represent the probability that a finite set $A \subset \mathcal{A}$ is the set of agents. The probability that n agents will participate in the auction is $\gamma_A(n) = \sum_{A,|A|=n} \beta_A$. All agents know the probability distribution β_A. Once an agent k is selected, he updates his probability of the number of agents present as:

$$p_n^k = \frac{\sum_{A,|A|=n,k\in A} \beta_A}{\sum_{A,k\in A} \beta_A}. \tag{2}$$

We deviate from the model in [10] by adding the assumption that all bidders are equally likely to be chosen. Hence p_n^k is the same for all k; we will hereafter refer only to p_n. Finally, we assume that $\gamma_A(0) = \gamma_A(1) = 0$; at least two agents will participate in the auction.

An equilibrium for this setting was demonstrated by Harstad, Kagel and Levin [5]:

Proposition 3 *If valuations are selected from a continuous distribution F and the number of bidders is selected from the distribution P then it is a unique symmetric equilibrium for each agent i to follow the strategy:*

$$b(v_i) = \sum_j \frac{F^{j-1}(v_i)p_j}{\sum_k F^{k-1}(v_i)p_k}\, b^e(v_i, j)$$

Observe that $b^e(v_i, j)$ is the amount of the equilibrium bid for a bidder with valuation v_i in a setting with j bidders as described in section 2.2 above. P is deduced from the economic environment.[2] We overload our previous notation for the equilibrium bid, this time as a function of the agent's valuation and the probability distribution P:

$$b^e(v_i, P) = \sum_j \frac{F^{j-1}(v_i)p_j}{\sum_k F^{k-1}(v_i)p_k}\, b^e(v_i, j) \tag{3}$$

Unfortunately, this auction model is still not rich enough to express our intuition about how agents could collude in a first-price auction. If each agent knows only a distribution on the total number of agents interested in participating in the auction, then he has no way of knowing that other agents have dropped out! It seems reasonable that agents will sometimes know how many agents are *placing bids* in the auction, even though they may not know the number of agents who chose not to participate at all. For example, when an auction takes place in an auction hall, no bidder knows how many potential bidders stayed home, but every bidder can count the number of people in the room before placing his or her bid. It is in this sort of auction that we could hope collusion based on dropping agents with low valuations would work. We must first introduce a new type of auction to model this auction hall scenario.

[2]Recall that P is a set: $p_j \in P$ for all $j \geq 0$, where p_j denotes the probability that the economic environment contains exactly j agents.

2.4 First-price auctions with participation revelation

First-price auctions with participation revelation are defined as follows:

1. Agents indicate their intention to bid in the auction.
2. The auctioneer announces n, the number of agents who registered in the first phase.
3. Agents submit bids to the auctioneer. The auctioneer will only accept bids from agents who registered in the first phase.
4. The agent who submitted the highest bid is awarded the good for the amount of his bid; all other agents are made to pay 0.

When a first-price auction with participation revelation operates in E_s, the equilibrium of the corresponding classical first-price auction holds.

Proposition 4 *In E_s it is an equilibrium of the first-price auction with participation revelation for every agent i to indicate the intention to participate and to bid according to $b^e(v_i, n)$.*

Proof. Agents are always better off participating in first-price auctions as long as there is no participation fee. The only way of participating is to declare the intention to participate in the first phase of the auction. Thus the number of agents announced by the auctioneer is equal to the total number of agents in the economic environment. From proposition 2 it is best for agent i to bid $b^e(v_i, n)$ when it is common knowledge that the number of agents in the economic environment is n. ■

Settings modelled using classical first-price auctions may often be more appropriately modelled as first-price auctions with participation revelation, since bidders rarely know *a priori* the number of opponents they will face. However, when bidders are unable to collude there is no strategic difference between these two mechanisms, explaining why the simpler classical model is commonly used. For the study of bidding clubs, however, the difference between the mechanisms is profound—we are now able to make the standard assumption that bidders have complete knowledge of the economic environment, while still finding that bidder strategies are affected by the number of other agents who indicate an intention to participate in the auction.

2.5 Distinguishing Features of our Model

Having justified our setting, it is worthwhile to emphasize the main differences between our model of collusion and models proposed in the work surveyed above (particularly [4] and [11]):

1. The number of bidders is stochastic.

2. There is no minimum number of bidders in a bidding club (e.g., bidding clubs are not required to contain all bidders).[3]

3. There is no limit to the number of bidding clubs in a single auction.

4. Club members and independent bidders behave strategically, acting according to correct beliefs about their environment.

Additionally, we make several restrictions on the bidding club protocols that we are willing to consider. None of these is required for the construction of a working protocol, but we feel that each of these characteristics is necessary for bidding clubs to be a realistic model of bidder cooperation:

1. Participation in bidding clubs requires an invitation, but bidders must be free to decline this invitation without (direct) penalty. In this way we include the choice to collude as one of agents' strategic decisions, rather than starting from the assumption that agents will collude.

2. Bidding club coordinators must make money on expectation. This ensures that third-parties have incentive to run bidding club coordinators.

3. The bidding club protocol must give rise to an equilibrium where all invited agents choose to participate, even when the bidding club operates in a single auction as opposed to a sequence of auctions. This means that agents can not be induced to collude in a given auction by the threat of being denied future opportunities to collude.

2.6 Overview

Section 3 expands the auction models and economic environments described above to the bidding club setting. Section 4 examines bidding club protocols for first-price auctions. After giving assumptions and two lemmas, we give a bidding club protocol for first-price auctions with participation revelation. Our main technical results are that:

- It is an equilibrium for agents to accept invitations to join bidding clubs when invited and to disclose their true valuations to their bidding club's coordinator, and for singleton agents to bid as they would in an auction with a stochastic number of participants in an economic environment without bidding clubs, in which the distribution over the number of participants is the same as in the bidding clubs setting.

- In equilibrium each agent is better off as a result of his own club (that is, his expected payoff is higher than would have been the case if his club never existed, but other clubs—if any—still did exist).

[3]For technical reasons we will have to assume that there is a finite *maximum* number of bidders in each bidding club; however, this maximum may be any integer greater than or equal to two.

- In equilibrium each club increases all non-members' expected payoffs, as compared to equilibrium in the case where all club members participated in the auction as singleton bidders, but all other clubs—if any—still existed.
- In equilibrium each agent is either better off or equally well off belonging to a bidding club as compared to equilibrium in the case where no clubs exist.

Finally, section 5 touches on questions of trustworthiness of coordinators, legality of bidding clubs and steps an auctioneer could take to disrupt the operation of bidding clubs.

3 Bidding Club Auction Model

In this section we extend both the economic environment and auction mechanism described above to include the characteristics necessary for a model of bidding clubs. As described above, our aim is not to model a situation where agents' *decision* to collude is exogenous, as this would gloss over the question of whether the collusion is stable. We thus include the collusive protocol as part of the model and show that it is individually rational *ex post* (i.e., after agents have observed their valuations) for agents to choose to collude. However, we do consider exogenous the selection of the set of agents who are *offered* the opportunity to collude. Furthermore, we want to show the impact of the possibility of collusion upon non-colluding agents; indeed, even colluding agents must take into account the possibility that *other* groups of agents in the auction may also be colluding. Once we have defined the new economic environment and auction mechanism, a well-defined Bayesian game will be specified by every tuple of primary auction type, bidding club rules and distributions of agent types, number of agents and number of bidding clubs.

3.1 The Economic Environment

We extend the economic environment E_s to consist of a set of agents who have non-negative valuations for a good at auction, the distinguished agent 0 and a set of bidding club coordinators who do not value the good, but may invite agents to participate in a bidding club. We will denote the new economic environment E_{bc}. Intuitively, in E_{bc} an agent's belief update after observing the number of agents in his bidding club does not result in any change in the distribution over the number of *other* agents in the auction, because the number of agents in each bidding club is independent of the number of agents in every other bidding club.

3.1.1 Coordinators

Coordinators are not free to choose their own strategies; rather, they act as part of the mechanism for a subset of the agents in the economic environment. We select coordinators in a process analogous to the approach for exogenously selecting agents in [10]: we draw a finite set of individuals from an infinite set of potential coordinators. In this case, however, this finite set is considered "potential coordinators"; in section 3.1.2 we will describe which potential coordinators are "actualized", i.e., correspond to actual coordinators.

Let $\mathcal{C} \equiv \mathbb{N}$ (excluding 0) be the set of all coordinators. β_C represents the probability that a finite set $C \subset \mathcal{C}$ is selected to be the set of potential coordinators. We add the restriction that all coordinators are equally likely to be chosen. A consequence of this restriction is that an agent's knowledge of the coordinator with whom he is associated does not give him additional information about what other coordinators may have been selected. We denote the probability that an auction will involve n_c potential coordinators as $\gamma_C(n_c) = \sum_{C,|C|=n_c} \beta_C$. We assume that $\gamma_C(0) = \gamma_C(1) = 0$: at least two potential coordinators will be associated with each auction.

3.1.2 Agents

We independently associate a random number of agents with each potential coordinator, again drawing a finite set of actual agents from an infinite set of potential agents. If only one (actual) agent is associated with a potential coordinator, the potential coordinator will not be actualized and hence the agent will not belong to a bidding club. In this way we model agents who participate directly in the auction without being associated with a coordinator. If more than one agent is associated with a potential coordinator, the coordinator *is* actualized and all its associated agents receive an invitation to participate in the bidding club.

Let $\mathcal{A} \equiv \mathbb{N}$ be the set of all agents, and let $\kappa \in \mathbb{N} \setminus \{0, 1\}$ be the maximum number of agents who may be associated with a single bidding club. Partition $\mathcal{A}$ into subsets, where agent i belongs to the subset $\mathcal{A}_{\lceil i/\kappa \rceil}$. Let β_A be the probability that a finite set $A \subset \mathcal{A}_i$ is the set of agents associated with potential coordinator i; we assume that all agents are equally likely to be chosen. The probability that n agents will be associated with a potential coordinator is denoted $\gamma_A(n) = \sum_{A,|A|=n} \beta_A$. By the definition of κ, $\forall j > \kappa, \gamma_A(j) = 0$; we assume that $\gamma_A(0) = 0$ and that $\gamma_A(1) < 1$.

3.1.3 Types and Signals

Recall that the type $\tau_i \in \mathcal{T}$ of agent i is the pair $(v_i, s_i) \in V \times \mathcal{S}$. Let $\mathcal{S} \in \mathbb{N} \setminus \{0\}$; s_i denotes agent i's private information about the number of agents in his bidding club.[4] Of course, if this number is 1 then there is no coordinator for the agent to deal with, and he will simply participate in the main auction. Note also that agents are neither aware of the number of potential coordinators

[4] In fact, none of our results require that agents know the *number* of agents in their bidding clubs; it would be sufficient that agents know *whether* they belong to a bidding club. We consider the setting where agents' signals are more informative because deviation from the bidding club protocol is *more* profitable in this case.

for their auction nor the number of actualized potential coordinators, though they are aware of both distributions.

3.1.4 Beliefs

Once an agent is selected, he updates his probability distribution over the number of actual agents in the economic environment. Not all agents will have the same beliefs—agents who have been signaled that they belong to a bidding club will expect a larger number of agents than singleton agents. We denote by $p_m^{n,k}$ the probability that there are a total of m agents in the auction, given that there are n bidding clubs and that there are k agents in the bidder's own club; we denote the whole distribution $P^{n,k}$. Because the numbers of agents in each bidding club are independent, observe that every agent in the whole auction has the same beliefs about the number of other agents in the economic environment, discounting those agents in his own bidding club. Hence agent i's beliefs are described by the distribution P^{n,s_i}.

3.2 The Augmented Auction Mechanism

Bidding clubs, in combination with a main auction, induce an augmented auction mechanism for their members:

1. A set A of bidders is invited to join the bidding club.
2. Each agent i sends a message μ_i to the bidding club coordinator. This may be the null message, which indicates that i will not participate in the coordination and will instead participate freely in the main auction. Otherwise, i agrees to be bound by the bidding club rules, and μ_i is i's declared valuation for the good. Of course, i can lie about his valuation.
3. Based on commonly-known rules and the information all the members supply, the coordinator selects a subset of the agents to bid in the main auction. We assume that the coordinator can force agents to bid as desired, e.g. by imposing a punitive charge on misbehaving agents.
4. The coordinator makes a payment to each club member. The amount of the payment must not depend on any of the agents' declared valuations or on the outcome of the main auction.
5. If a bidder represented by the coordinator wins the main auction, he is made to pay the amount required by the auction mechanism to the auctioneer. In addition, he may be required to make an additional payment to the coordinator.

Any number of coordinators may participate in an auction. However, we assume that there is only a single coordination protocol, and that this protocol is common knowledge.

4 Bidding Clubs for First-Price Auctions

This section contains the paper's main technical results. We begin by stating some (mild) assumptions about the distribution of agent valuations, then use these assumptions to prove a technical lemma. A second lemma explains how we can show the existence of an equilibrium in a setting where agents receive asymmetric information and are subject to asymmetric payment rules. We then give the bidding club protocol for first-price auctions, based on a first-price auction with participation revelation as described in section 2.4. We show an equilibrium of this auction, and demonstrate that agents gain under this equilibrium.

4.1 Assumptions about F

Our results hold for a broad class of distributions of agent valuations—all distributions for which the following two assumptions are true.

Assumption 1 *F is continuous and atomless.*

In order to give our second assumption, we must introduce some notation:

$$P_{x\geq i} = \sum_{x=i}^{\infty} p_x. \tag{4}$$

We now define the relation "<" for probability distributions:

$$P < P' \text{ iff } \exists l(\forall i < l, P_{x\geq i} = p'_{x\geq i} \text{ and } \forall i \geq l, P_{x\geq i} < P'_{x\geq i}). \tag{5}$$

We are now able to state our second assumption:

Assumption 2 *$(P < P')$ implies that $\forall v$, $b^e(v, P) < b^e(v, P')$*

Intuitively, we assume that every agent's symmetric equilibrium bid in E_s with number of participants drawn from P' is strictly greater than that agent's symmetric equilibrium bid in E_s with number of participants drawn from P, in the case where P' stochastically dominates P.[5]

4.2 A Technical Lemma

It is important to note that the notation $P^{n,k}$ may be seen as defining a probability distribution over the number of agents in economic environment E_s (i.e., even without the existence of bidding clubs). It is thus possible to discuss equilibrium bids in the classical stochastic settings where the number of bidders is drawn from such a distribution. While it will remain to show why these values are meaningful in our setting where (among other differences) agents have

[5]This assumption holds for every standard distribution of independent valuations of which we are aware.

asymmetric information, it will be useful to prove the following lemma about the classical stochastic setting:[6]

Lemma 1 $\forall k \geq 2, \forall n \geq 2, \forall v,\ b^e(v, P^{n+k-1,1}) > b^e(v, P^{n,k})$

Remark. This lemma asserts that the symmetric equilibrium bid is always higher when more agents belong to the main auction as singleton bidders and the total number of agents is held constant.

Proof. Recall Assumption 2 from section 4.1. We defined $P < P'$ as the proposition that $\exists l(\forall i < l, P_{x \geq i} = P'_{x \geq i}$ and $\forall i \geq l, P_{x \geq i} < P'_{x \geq i})$, and assumed that $(P < P')$ implies that $\forall v, b^e(v, P) < b^e(v, P')$. It is thus sufficient to show that $P^{n+k-1,1} > P^{n,k}$. We will take $l = n + k$.

First we will show that $\forall j < n + k, P^{n+k-1,1}_{x \geq j} = P^{n,k}_{x \geq j}$. The distribution $P^{n+k-1,1}$ expresses the belief that there are $n+k-2$ potential coordinators, the membership of which is distributed as described in section 3.1, and one potential coordinator that is known to contain only a single bidder. The distribution $P^{n,k}$ expresses the belief that there are $n-1$ potential coordinators, the membership of which is again distributed as described in section 3.1, and one potential coordinator that is known to contain exactly k bidders. Under both distributions it is certain that there are at least $n + k - 1$ agents. Therefore $\forall j < n + k, P^{n+k-1,1}_{x \geq j} = P^{n,k}_{x \geq j} = 1$.

Second, $\forall j \geq n + k$, $P^{n+k-1,1}_{x \geq j} > P^{n,k}_{x \geq j}$. Considering $P^{n+k-1,1}$, observe that for $n + k - 2$ of the potential coordinators the probability that this coordinator contains a single agent is less than one and these probabilities are all independent; the last potential coordinator contains a single agent with probability one. Considering $P^{n,k}$, there are $n - 1$ potential coordinators where the probability of containing a single agent is less than one, exactly as above, and k potential coordinators certain to contain exactly one agent. Thus the two distributions agree exactly about $n - 1$ of the potential coordinators, which both hold to contain more than a single agent, and likewise both distributions agree that one of the potential coordinators contains exactly one agent. However, there remain $k - 1$ potential coordinators about which the distributions disagree; $P^{n+k-1,1}$ always generates a greater or equal number of agents for these potential coordinators, as compared to $P^{n,k}$. Under the latter distribution all these agents are singletons with probability one, while under the former there is positive probability that each of the potential coordinators contains more than one agent. As long as $k \geq 2$, there is at least one potential coordinator for which $P^{n+k-1,1}$ stochastically dominates $P^{n,k}$. Thus $\forall k \geq 2, \forall n \geq 2, \forall v\, P^{n+k-1,1} > P^{n,k}$. ■

[6]For convenience and to preserve intuition in what follows we will refer to the number of potential coordinators and the number of agents belonging to a coordinator even though we concern ourselves with the economic environment E_s where bidding clubs do not exist. The number of potential coordinators is shorthand for the number n_c drawn from γ_C in the first phase of the procedural definition of the distribution $P^{n,k}$. Likewise the number of agents associated with a potential coordinator is shorthand for the number of agents chosen from one of the n_c iterative draws from γ_A.

4.3 Truthful Equilibria in Asymmetric Mechanisms

In E_{bc} there is informational asymmetry because agents receive different signals, and asymmetric payment rules because some agents belong to bidding clubs of different sizes and others do not belong to a bidding club at all. The lemma in this section will allow us to go on to show an equilibrium in Theorem 1 despite these asymmetries.

We describe a particular class of auction mechanisms that are asymmetric in the sense that every agent is subject to the same allocation rule but to a potentially different payment rule, and furthermore that agents may receive different signals. A truth-revealing equilibrium exists in such auctions when the following conditions hold:

1. The auction allocates the good to the agent who submits the highest bid.
2. Consider the auction M_i in which *all* agents are subject to agent i's payment rule and the above allocation rule, and where (hypothetically) *all* agents receive the signal s_i.[7] Truth-revelation is a symmetric equilibrium in M_i.

Observe that the second condition above is less restrictive than it may appear. From the revelation principle we can see that for every auction with a symmetric equilibrium there is a corresponding auction in which truth-revealing is an equilibrium that gives rise to the same allocation and the same payments for all agents. M_i can thus be seen as a revelation mechanism for any other auction that has a symmetric equilibrium.

Definition 1 $\bar{M}$ *is a* regular asymmetric *auction if it has the following structure, where* $\boldsymbol{M}$ *represents a set of auctions* $\{M_1, \ldots, M_n\}$ *which each allocate the good to the agent who submits the highest bid, and which are all truth-revealing direct mechanisms for* n *risk-neutral agents with independent private valuations drawn from the same distribution:*

1. *Each agent* i *sends a message* μ_i *to the center.*
2. *The center allocates the good to the agent* i *with* $\mu_i \in \max_j \mu_j$*. If multiple agents submit the highest message, the tie is broken in some arbitrary way.*
3. *Agent* i *is made to transfer* $t_i(\mu, \pi)$ *to the center. The transfer function* t_i *is taken from* $M_i \in \boldsymbol{M}$*.*

Lemma 2 *Truth-revelation is an equilibrium of regular asymmetric auctions.*

Proof. The payoff of agent i is uniquely determined by the allocation rule, the transfer function t_i, and all agents' strategies. Assume that the other agents are truth revealing, then the other agents' behavior, the allocation rule, and

[7] That is, for every agent j in the real auction, we create an agent k in the hypothetical auction M_i having type $\tau_k = (v_j, s_i)$.

agent i's payment rule are all identical in $\bar{M}$ and M_i. Since truth-revelation is an equilibrium in M_i, truth-revelation is agent i's best response in $\bar{M}$. ■

The next corollary, following directly from Lemma 2, compares a single agent's expected utility under two different auctions which implement different payment rules. We will need this result for our proof of Theorem 1.

Corollary 1 *Consider two regular asymmetric auctions $\bar{M}$ and $\bar{M}'$, which both implement the same transfer function for agent i. In equilibrium, agent i's expected utility is the same in both $\bar{M}$ and $\bar{M}'$.*

Proof. The payoff of agent i is uniquely determined by the allocation rule, its transfer function, and all agents' strategies. Both $\bar{M}$ and $\bar{M}'$ have the same allocation rule. Lemma 2 tells us that truth revelation is a best response for all agents in both $\bar{M}$ and $\bar{M}'$, so all agents' strategies are identical in the two auctions. In general, agents may not receive the same expected utility from $\bar{M}$ and $\bar{M}'$. However, since i has the same transfer function in both auctions, i's expected utility in $\bar{M}$ is equal to his expected utility in $\bar{M}'$. ■

4.4 First-Price Auction Bidding Club Protocol

What follows is the protocol of a coordinator who approaches k agents.

1. Each agent i sends a message μ_i to the coordinator.
2. If at least one agent declines participation then the coordinator registers in the main auction for every agent who accepted the invitation to the bidding club. For each bidder i, the coordinator submits a bid of $b^e(\mu_i, P^{n,k})$, where n is the number of bidders announced by the auctioneer.
3. If all k agents accepted the invitation then the coordinator drops all bidders except the bidder with the highest reported valuation, who we will denote as bidder h. For this bidder the coordinator places a bid of $b^e(\mu_h, P^{n,1})$ in the main auction.
4. The coordinator pays each member a pre-determined payment $c \geq 0$ whenever all bidders participate in the club, and regardless of the outcome of the auction and of how much each bidder bid. Following the argument in [3] let g be the coordinator's *ex ante* expected gain if all agents behave according to the equilibrium in Theorem 1; the coordinator will not lose money on expectation if it pays each agent $c = \frac{1}{k}(g - c')$ with $0 \leq c' \leq g$.
5. If bidder h wins in the main auction, he is made to pay $b^e(\mu_h, P^{n,1})$ to the center and $b^e(\mu_h, P^{n,k}) - b^e(\mu_h, P^{n,1})$ to the coordinator.

Observe that in equilibrium the coordinator has an expected profit of c', though it will lose kc whenever the winner of the main auction does not belong to its club. If a coordinator wanted to be budget-balanced on expectation rather than profitable on expectation, it could set $c' = 0$.

We are now ready to prove the main theorem of the paper:

Theorem 1 *It is an equilibrium for all bidding club members to choose to participate and to truthfully declare their valuations to their respective bidding club coordinators, and for all non-bidding club members to participate in the main auction with a bid of* $b^e(v, P^{n,1})$.

Proof. We first prove that the above strategy is in equilibrium for both categories of bidders assuming that agents all participate; we then prove that participation is rational for all agents.

For the proof of equilibrium we consider a one-stage mechanism which behaves as follows:

1. The center announces n, the number of bidders in the main auction.
2. Bidders submit bids (messages) to the mechanism.
3. The bidder with the highest bid is allocated the good.
4. The winning bidder is made to pay $b^e(v_i, P^{n,s_i}) - c$.
5. All non-winning bidding club members are paid c.

This one-stage mechanism has the same payment rule for bidding club bidders as the bidding club protocol given above, but no longer implements a first-price payment rule for singleton bidders. In order to prove that the strategies given in the statement of the theorem are an equilibrium, it is sufficient to show that truthful bidding is an equilibrium for all bidders under the given one-stage mechanism. Observe that this mechanism may be seen as a mechanism $\bar{M}$ in the sense of Lemma 2: it allocates the good to the agent who submits the highest message, and (by definition of b^e) the auction M_i in which *all* agents are subject to agent i's payment rule and receive the signal s_i has truth revelation as a symmetric equilibrium.

Strategy of non-club bidder: Assume that all bidding club agents (if any) bid truthfully. Further assume that all non-club agents also bid truthfully except for non-club bidder i. The probability distribution $P^{n,1}$ correctly describes the distribution of the number of agents faced by i, given his signal $s_i = 1$ and the auctioneer's announcement that there are n bidders in the main auction. Although agents in bidding clubs have additional information about the number of agents—each agent knows that there is at least one other agent in his own club—their prescribed behavior is to place bids of $b^e(\mu, P^{n,1})$ in the main auction. Agent i thus faces an unknown number of agents distributed according to $P^{n,1}$ and all bidding $b^e(v, P^{n,1})$. The auction is regular asymmetric: using the result from Lemma 2, i's strategic decision is the same as under a mechanism where all agents are subject to his payment rule and share his signal s_i, and with a stochastic number of bidders distributed according to $P^{n,1}$. In particular, it does not matter that the club members are subject to different payment rules and have additional information, and so i will also bid $b^e(v, P^{n,1})$.

Strategy of club bidder: Assume that all agents accept the invitation to join their respective clubs and then truthfully declare their valuations, excluding club bidder i who decides to participate but considers his bid. Once again, observe that the auction is regular asymmetric, and so Lemma 2 applies: $P^{n,k}$ describes the distribution over the number of agents conditioned on i's signal $s_i = k$, and the bidder submitting the highest (global) message will always be allocated the good. Therefore truthful bidding is a best response for agent i, despite the information asymmetry. Because i gets the payment c regardless of the amount of his bid, the presence or absence of this payment has no effect on his choice of what amount to bid given the decision to participate.

We now turn to the question of participation; for this part of the proof we consider the original, multi-stage mechanism.

Participation of non-club bidder: Because there is no participation fee, it is always rational for a bidder to participate in a first-price auction.

Participation of club bidder: Assume that $c = 0$; clearly $c > 0$ only increases agents' incentive to participate in a bidding club. Because there is no participation fee, all bidding club bidders will participate in the auction, but must decide whether or not to accept their coordinators' invitations. Assume that all agents except for i join their respective clubs and bid truthfully, and agent i must decide whether or not to join his bidding club. Agent i knows the number of agents in his bidding club and updates his distribution over the number of agents in the whole auction as $P^{n,k}$.

Consider the classical stochastic case where all bidders have the same information as i (and are subject to the same payment rules): from proposition 3 it is a best response for i to bid $b^e(v_i, P^{n,k})$. In this setting i's expected gain is the same as in the equilibrium of the one-stage mechanism from the first part of the proof where all bidding club members (including i) join their clubs and bid truthfully (with $c = 0$), by Corollary 1.

As a result of i declining the offer to participate in the bidding club there are $n - 1$ bidders in the main auction placing bids of $b^e(v, P^{n+k-1,1})$ and $k - 1$ other bidders placing bids of $b^e(v, P^{n,k})$. We know from Lemma 1 that $b^e(v, P^{n+k-1,1}) > b^e(v, P^{n,k})$. Thus the singleton bidders and other bidding clubs will bid a higher function[8] of their valuations than the bidders from the disbanded bidding club. It always reduces a bidder's expected gain in a first-price auction to cause other bidders to bid above the equilibrium, because it reduces the chance that he will win without affecting his payment if he does win. This is exactly the effect of i declining the offer to join his bidding club: the $k - 1$ other bidders from i's bidding club bid according to the equilibrium of the classical stochastic case discussed above, but the $n-1$ singleton and bidding club bidders submit bids that exceed the symmetric equilibrium amount. Therefore i's expected gain is smaller if he declines the offer to participate than if he accepts it. ■

[8]Note that this occurs because the singleton bidders and other bidding clubs in the main auction follow a strategy that depends on the number of bidders announced by the auctioneer; hence they bid as though all the $k - 1$ bidders from the disbanded bidding club might each be independent bidding clubs.

4.5 Do bidding clubs cause agents to gain?

All things being equal, bidders are better off being invited to a bidding club than being sent to the auction as singleton bidders. Intuitively, an agent gains by not having to consider the possibility that other bidders who would otherwise have belonged to his bidding club might themselves be bidding clubs.

Theorem 2 *An agent i has higher expected utility in a bidding club of size k bidding as described in Theorem 1 than he does if the bidding club does not exist and k additional agents (including i) participate directly in the main auction as singleton bidders, again bidding as described in Theorem 1, for $c \geq 0$.*

Proof. Consider the counterfactual case where agent i's bidding club does not exist, and all the members of this bidding club are replaced by singleton bidders in the main auction. We will show that i is better off as a member of the bidding club (even when $c = 0$) than in this case. If there were n potential coordinators in the original auction and k agents in i's bidding club, then the auctioneer would announce $n+k-1$ as the number of participants in the new auction. Under the equilibrium from Theorem 1, as a singleton bidder i will bid $b^e(v_i, P^{n+k-1,1})$. If he belonged to the bidding club and followed the same equilibrium i would bid $b^e(v_i, P^{n,k})$. In both cases the auction is economically efficient, which means i is better off in the auction that requires him to pay a smaller amount when he wins. Lemma 1 shows that $\forall k \geq 2, \forall n \geq 2, \forall v, b^e(v, P^{n+k-1,1}) > b^e(v, P^{n,k})$, and so our result follows. ■

We can also show that singleton bidders and members of other bidding clubs benefit from the existence of each bidding club in the same sense. Following an argument similar to the one in Theorem 2, other bidders gain from not having to consider the possibility that additional bidders might represent bidding clubs. Paradoxically, as long as $c' > 0$, other bidders' gain from the existence of a given bidding club is greater than the gain of that club's members.

Corollary 2 *In the equilibrium described in Theorem 1, singleton bidders and members of other bidding clubs have higher expected utility when other agents participate in a given bidding club of size $k \geq 2$, as compared to a case where k additional agents participate directly in the main auction as singleton bidders.*

Proof. Consider a singleton bidder in the first case, where the club of k agents does exist. (It is sufficient to consider a singleton bidder, since other bidding clubs bid in the same way as singleton bidders.) Following the equilibrium from Theorem 1 this agent would submit the bid $b^e(v_i, P^{n,1})$. Theorem 2 shows that it is better to belong to a bidding club (and thus to bid $b^e(v_i, P^{n,k})$) than to be a singleton bidder in an auction with the same number of agents (and thus to bid $b^e(v_i, P^{n+k-1,1})$. Since the distribution $P^{n,k}$ is just $P^{n,1}$ with $k-1$ singleton agents added, $\forall k \geq 2, b^e(v_i, P^{n,1}) < b^e(v_i, P^{n,k})$. Thus $\forall k \geq 2, b^e(v_i, P^{n,1}) < b^e(v_i, P^{n+k-1,1})$. ■

Finally, we can show that agents prefer participating in E_{bc} in the equilibrium from Theorem 1 in a bidding club of size k (thus, where the number of agents is distributed according to $P^{n,k}$) to participating in E_s with number of bidders distributed according to $P^{n,k}$, as long as $c > 0$.

Theorem 3 *For all $\tau_i \in \mathcal{T}$, for all $k \geq 2$, for all $n \geq 2$, for all $c > 0$, agent i obtains smaller expected utility by:*

1. *participating in a first-price auction with participation revelation in E_s with number of bidders distributed according to $P^{n,k}$; than by*
2. *participating in a bidding club of size k in E_{bc} and following the equilibrium from Theorem 1.*

When $c = 0$, agent i obtains the same expected utility in both cases.

Proof. For any efficient auction, an agent i's expected utility EU_i is $\sum_j P_j F^{j-1}(V_i)b$, where P_j is the probability that there are a total of j agents in the economic environment, $F^{j-1}(v_i)$ is the probability that i has the high valuation among these j agents, and b is the amount of i's bid.

First, we consider case (1). From proposition 4 it is an equilibrium for agent i in economic environment E_s to bid $b^e(v_i, j)$ in a first-price auction with participation revelation, where j is the number of bidders announced by the auctioneer. Since the number of agents is distributed according to $P^{n,k}$, agent i's expected utility in a first-price auction with participation revelation is:

$$\begin{aligned}
EU_{i,pr} &= \sum_j p_j^{n,k} F^{j-1}(v_i) b^e(v_i, j) &(6)\\
&= \frac{\sum_\ell p_\ell^{n,k} F^{\ell-1}(v_i)}{\sum_{\ell'} p_{\ell'}^{n,k} F^{\ell'-1}(v_i)} \sum_j p_j^{n,k} F^{j-1}(v_i) b^e(v_i, j)\\
&= \sum_\ell p_\ell^{n,k} F^{\ell-1}(v_i) \left(\sum_j \frac{p_j^{n,k} F^{j-1}(v_i)}{\sum_{\ell'} p_{\ell'}^{n,k} F^{\ell'-1}(v_i)} b^e(v_i, j) \right)\\
&= \sum_\ell p_\ell^{n,k} F^{\ell-1}(v_i) b^e(v_i, P^{n,k}) = EU_{i,s} &(7)
\end{aligned}$$

Equation (7) is agent i's expected utility in a first-price auction with a stochastic number of participants, which we shall denote $EU_{i,s}$. Observe that we make use of the definition of $b^e(v_i, P)$ from equation (3).

We now consider case (2). Let $EU_{i,bc}$ denote agent i's expected utility in E_{bc} as a member of a bidding club of size k, in the equilibrium from Theorem 1. Recall that in this equilibrium the bidder with the globally highest valuation always wins, and that all agents in bidding clubs of size k bid $b^e(v_i, P^{n,k})$ and receive a positive payment of c, which does not depend on the amount of their bids or on whether any agent in the club wins the auction.

$$EU_{i,bc} = \sum_j p_j^{n,k} F^{j-1}(v_i) b^e(v_i, P^{n,k}) + c \tag{8}$$

Intersecting equations (7) and (8), we get:

$$EU_{i,bc} - EU_{i,pr} = c \tag{9}$$

When $c > 0$, agent i's expected utility is strictly greater in case (2) than in case (1); when $c = 0$ he has the same expected utility in both cases. ■

What about agents who do not belong to bidding clubs? We can show in the same way that they are not harmed by the existence of bidding clubs: they are neither better nor worse off in the bidding club economic environment than facing the same distribution of opponents in a first-price auction with participation revelation.

Corollary 3 *For all $\tau_i \in \mathcal{T}$, for all $n \geq 2$, agent i obtains the same expected utility by:*

1. *participating in a first-price auction with participation revelation in E_s with number of bidders distributed according to $P^{n,1}$; as by*
2. *participating as a singleton bidder in E_{bc} and following the equilibrium from Theorem 1.*

Proof. We follow the same argument as in Theorem 3, except that $k = 1$ and $EU_{i,bc}$ does not include c. Thus we get $EU_{i,bc} = EU_{i,pr}$. ■

5 Discussion

In this section we consider the trustworthiness and legality of coordinators, and also discuss two ways for auctioneers to disrupt bidding clubs in their auctions.

5.1 Trust

Why would a bidding club coordinator be willing to provide reliable service, and likewise why would bidders have reason to trust a coordinator? For example, a malicious coordination protocol could be used simply to drop all its members from the auction and reduce competition. While this is a reasonable concern, our coordinators make a profit on expectation, thus providing incentive for a trusted third party to run a reliable coordination service. Indeed, coordinators would be very inexpensive to run: as their behavior is entirely deterministic, they could operate without any human supervision. The establishment of trust is exogenous to our model; we have simply assumed that all agents trust coordinators and that all coordinators are honest.

5.2 Legality

We have often been asked about the legal issues surrounding the use of bidding clubs. While this is an interesting and pertinent question, it exceeds both our expertise and the scope of this paper. We should note, however, that uses of bidding clubs exist that might not fall under the legal definition of collusion. For example, a corporation could use a bidding club to choose one of its departments to bid in an external auction. In this way the corporation could be sure to avoid bidding against itself in the external auction while avoiding dictatorship and respecting each department's self-interest. Coordinators may also be permitted by the auctioneer: e.g., by an internet market seeking to attract more bidders to its site.

5.3 Disrupting Bidding Clubs

There are two things an auctioneer can do to disrupt bidding clubs in a first-price auction. First, she can permit "false-name bidding." (Our auction model has assumed that each agent may place only a single bid in the auction, and that the center has a way of uniquely identifying agents.) Second, she can refrain from publicly disclosing the winner of the auction.

If bidders can bid both in their bidding clubs and in the main auction, they are better off deviating from the equilibrium in Theorem 1 in the following way. A bidder i can accept the invitation to join the bidding club but place a very low bid with the coordinator; at the same time, i can directly submit a competitive bid in the main auction. Agent i will gain by following this strategy when all other agents follow the strategies specified in Theorem 1 because accepting the invitation to join the bidding club ensures that the club does drop all but one of its members and also causes the high bidder to bid less than he would if he were not bound to the coordination protocol. If the bidding club drops any bidders other than i then all agents' bids will also be lowered because the number of participants announced by the auctioneer will be smaller, compared to the case where the bidding club did not exist or where it was disbanded. However, if false-name bidding is impossible and the winner of the auction is publicly disclosed then the bidding club coordinator can detect an agent who has deviated in this way. Because the agent has agreed to participate in the bidding club the coordinator has the power to punish this agent and make the deviation unprofitable. If either or both of these requirements does not hold, however, the coordinator will be unable to detect defection and so the equilibrium from Theorem 1 will not hold.

6 Conclusion

We have presented a formal model of bidding clubs which in many ways extends models traditionally used in the study of collusion; most importantly, all agents behave strategically based on correct information about the economic environment, including the possibility that other agents will collude. Other features of our setting include a stochastic number of agents and of bidding clubs in each auction, and revelation by the auctioneer of the number of bids received. The strategy space is expanded so that the decision of whether or not to join a bidding club is part of an agent's choice of strategy. Bidding clubs make money on expectation, and can optionally be configured so they never lose money. We have showed a bidding club protocol for first-price auctions that leads to a (globally) efficient allocation in equilibrium, and which does not make use of side-payments in the case of $c = 0$. There are three ways of asking the question of whether agents gain by participating in bidding clubs in first-price auctions:

1. Could any agent gain by deviating from the protocol?
2. Would any agent be better off if his bidding club did not exist?
3. Would any agent would be better off in an economic environment that did not include bidding clubs at all?

We have shown that agents are strictly better off in all three senses. (In the third sense, the gain is only strict when $c > 0$.) We have also shown that each bidding club causes *non-members* to gain in the second sense, and does not hurt them in the third sense. Finally, we have discussed ways for an auctioneer to set up the rules of her auction so as to disrupt the operation of bidding clubs.

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