On Nonholonomic Mobile Robots and Optimal Maneuvering

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Abstract: We consider the robot path planning problem in the presence of non-integrable kinematic constraints, known as nonholonomic constraints. Such constraints are generally caused by one or several rolling contacts between rigid bodies and express that the relative velocity of two points in contact is zero. They make the dimension of the space of achievable velocities smaller than the dimension of the robot's configuration space. Using standard results in differential geometry (Frobenius Integrability Theorem) and nonlinear control theory, we first give a formal characterization of holonomy (and nonholonomy) for robot systems subject to linear differential constraints and we state some related results about their controllability. Then, we apply these results to "car-like" robots and "trailer-like" robots. Finally, we present an implemented planner, which generates collision-free paths with minimal number of maneuvers for car-like and trailer-like robots among obstacles. Potential applications of the planner include navigation of autonomous robots, automated parking of personal cars and trucks, autonomous navigation of luggage carriers in airport facilities, automatic planning of the movements of machines in a construction site, and computer-aided design of access ports for trucks in industrial and commercial facilities.

Acknowledgements: This research was funded by DARPA contract DAAA21-89-C0002 (Army), CIS (Center for Integrated Systems), CIFE (Center for Integrated Facility Engineering), and Digital Equipment Corporation.

1 Introduction

We consider the robot path planning problem in the presence of non-integrable kinematic constraints, known as *nonholonomic* constraints [Greenwood, 1965]. Such constraints are generally caused by one or several *rolling contacts* between rigid bodies and express that the relative velocity of two points in contact is zero. They make the dimension of the space of achievable velocities smaller than the dimension of the robot's configuration space.

A car is a typical nonholonomic mechanical system. In the absence of obstacles, it can attain any position in the plane, with any orientation. Hence, the configuration space is three-dimensional. However, assuming no slipping, the velocity of the midpoint between the two rear wheels of the car is always tangent to the car orientation. The space of achievable velocities at any configuration is thus two-dimensional.

Collision-free path planning consists of constructing a path connecting two input configurations in the subset of configurations where the robot has no contact or intersection with the obstacles. Nonholonomic constraints require that the tangent to the path at any configuration be within the subspace of velocities selected out by the constraints. A collision-free path for a nonholonomic robot typically includes "maneuvers", i.e. backing-up points where the robot stops and changes the sign of the velocity. Finding a feasible path between two configurations is one difficult problem. Another one is to minimize the number of maneuvers, or at least to keep it reasonable, whenever possible.

The first part of the paper is a mathematical analysis of nonholonomic constraints. Using standard results in differential geometry and nonlinear control theory, we give a formal characterization of holonomy for robots subject to linear differential constraints and we state related results about their controllability. In particular, we establish two effective results applicable when the robot is subject to a single linear differential constraint. The first result allows us to determine through simple computation if this constraint is holonomic, or not. The second result states that any robot subject to a single linear nonholonomic equality constraint is fully controllable, that is: any two configurations lying in an open connected subset of the configuration space can be connected by a path lying in this subset and respecting the nonholonomic constraint.

The second part of the paper applies these results to two types of robotic systems, which we name "car-like robots" and "trailer-like robots". A car-like robot is kinematically similar to an automobile car. A trailer-like robot is kinematically similar to a vehicle towing a trailer.

Finally, in the third part of the paper, we present an implemented planner, which generates collision-free paths for car-like and trailer-like robots moving among obstacles. A version of the planner generates paths with minimal number of maneuvers in reasonable amount of time.

Research on collision-free path planning has been very active during the past ten years (e.g., see [Lozano-Pérez, 1983] [Schwartz, Hopcroft and Yap, 1987]). Today, the mathematical and computational structures of path planning for holonomic robots is reasonably well-understood. Practical planners have also been implemented in more or less specific cases (e.g., [Brooks and Lozano-Pérez, 1983] [Gouzènes, 1984] [Faverjon and Tournassoud, 1987] [Lozano-Pérez, 1987] [Barraquand, Langlois and Latombe, 1989]). However, path planning with nonholonomic constraints has attracted much less interest so far.

The problem was first introduced by Laumond [Laumond, 1986], who proved that a car-like robot is fully controllable, even when the steering angle is limited. (Our result on robot controllability with a single nonholonomic constraint is a generalization of Laumond's result, since it is established for any nonholonomic linear equality constraint. But, it considers no such constraint as limited steering angle, which is a nonholonomic nonlinear inequality constraint.) However, the number of maneuvers that would be generated by a planner implementing the constructive proof of this result is unbounded, even when there exists collision-free paths with no or few maneuvers that satisfy the nonholonomic constraint.

Building on his previous work, Laumond proposed an algorithm for planning smooth – i.e., maneuver-free – collision-free paths of a nonholonomic circular robot whose turning radius is lower-bounded [Laumond, 1987]. However, this interesting algorithm fails whenever all free paths require one or more maneuvers. Tournassoud and Jehl proposed a specific technique for planning paths with simple maneuvers for a car-like robot turning in a corridor [Tournassoud and Jehl, 1988]. They also suggested a generalization of this result by decomposing the empty subset of the workspace into corridor-like regions. Li and Canny first pointed out that results in nonlinear control theory where transposable in order to characterize the controllability of nonholonomic robots [Li and Canny, 1989].

The planner presented in this paper is essentially the planner described in detail in [Barraquand and Latombe, 1989]. It makes use of a discretized representation of the workspace and the configuration space. We ran several experiments with it, using simulated car-like and trailer-like robots with obstacle arrangements requiring backing-up maneuvers. The experiments reported in this paper were carried out with a version of the planner specifically designed to optimize the number of maneuvers. In this version, the planner applies a brute force method that consists of performing a dynamic search in the discretized configuration space with the number of maneuvers as the cost function. Despite its conceptual simplicity, the planner is fast in non-trivial workspaces. To our knowledge, this is the first implemented planner capable of finding collision-free path with minimal number of maneuvers (at the resolution of the configuration space representation) for nonholonomic robots.

Possible applications of the planner include navigation of autonomous robots, automated parking of personal cars and trucks, autonomous navigation of luggage carriers in airport facilities, automatic planning of the movements of machines in a construction site, and computer-aided design of access ports for trucks in industrial and commercial facilities.

2 Nonholonomic Constraints

2.1 Terminology

We denote by $\mathcal{A}$ the robot and $\mathcal{W}$ its workspace. A **configuration** of $\mathcal{A}$ is a specification of the position of every point in $\mathcal{A}$ with respect to a Cartesian frame embedded in $\mathcal{W}$. The **configuration space** of $\mathcal{A}$ is the space, denoted by $\mathcal{C}$, of all the possible configurations of $\mathcal{A}$. The configuration space of a mechanical system made of rigid bodies is a smooth manifold [Arnold, 1978]. For instance, the configuration space of a two-dimensional rigid body translating and rotating in $\mathcal{W} = \mathbf{R}^2$ is $\mathcal{C} = \mathbf{R}^2 \times S^1$, where S^1 denotes the unit circle. In virtually any practical situation, the range of positions reachable by the robot's bodies can be bounded, making $\mathcal{C}$ into a compact manifold.

In the following, we represent a configuration q of $\mathcal{A}$ by a list of n parameters, $(q_1, q_2, ..., q_n)$, where n is the dimension of $\mathcal{C}$. This representation corresponds to defining an *atlas* of $\mathcal{C}$. Each configuration q belongs to at least one neighborhood covered by a *chart* of the atlas. The parameters $q_1, ..., q_n$ are the *coordinates* of q in this chart. These parameters are also called *generalized coordinates* of $\mathcal{A}$ [Greenwood, 1965]. For instance, we will represent the configuration of a car-like robot by $\mathrm{q} = (X_f, Y_f, \theta)$, where X_f and Y_f are the coordinates of the midpoint between the two front wheels of the car in the Cartesian frame embedded in $\mathcal{W}$ and θ is the orientation of the main axis of the robot relatively to the x axis of this Cartesian frame. Obviously, there is not a unique set of generalized coordinates for a given robot. However, the various charts put on a smooth manifold are C^∞-*related*, which allows to extend differential properties established in a chart – i.e. with a generalized coordinate system – to all the other charts.

Now, suppose that a scalar constraint of the form:

$$F(\mathrm{q}, t) = 0 \tag{1}$$

with $\mathrm{q} \in \mathcal{C}$ and t denoting time, applies to the motion of $\mathcal{A}$. Let us further assume that F is smooth with non-zero derivative. In theory, one could use the equation to solve for one of the generalized coordinates in terms of the other coordinates and time. Thus, equation (1) defines a $(n-1)$-dimensional submanifold of $\mathcal{C}$. This submanifold is in fact the actual configuration space[1] of $\mathcal{A}$ and the $n-1$ remaining coordinates its actual generalized coordinates. Constraint (1) is called a **holonomic** equality constraint. More generally, there may be k constraints of the form (1). If they are independent – i.e., their Jacobian matrix has full rank – they determine a $(n-k)$-dimensional submanifold of $\mathcal{C}$, which is the actual configuration space of $\mathcal{A}$.

A constraint of the form $F(\mathrm{q}, t) < 0$ or $F(\mathrm{q}, t) \leq 0$ where F is smooth with non-zero derivative, is a holonomic inequality constraint. It typically acts as a mechanical stop or an obstacle. It simply determines a submanifold of $\mathcal{C}$ having the same dimension as $\mathcal{C}$.

Constraint (1) is only a kinematic constraint of one sort. Now, suppose that a scalar constraint of the form:

$$G(\mathrm{q}, \dot{\mathrm{q}}, t) = 0 \tag{2}$$

applies to the motion of $\mathcal{A}$, with $\dot{\mathrm{q}} \in T_{\mathrm{q}}(\mathcal{C})$, the *tangent space* of $\mathcal{C}$ at q. The tangent space represents the space of the velocities of $\mathcal{A}$. $T_{\mathrm{q}}(\mathcal{C})$ has dimension n for every $\mathrm{q} \in \mathcal{C}$.

A kinematic constraint of the form (2) is holonomic if it is integrable, i.e. $\dot{\mathrm{q}}$ can be eliminated and the equation (2) rewritten in the form (1). Otherwise, the constraint is called a **nonholonomic** equality constraint. As we will see below, a nonholonomic equality constraint restricts the space of velocities achievable by $\mathcal{A}$ at any configuration q to a $(n-1)$-dimensional linear subspace of $T_{\mathrm{q}}(\mathcal{C})$, without affecting the dimension of the configuration space. If there are k independent nonholonomic equality constraints of the form (2), the space of achievable velocities is a subspace of $T_{\mathrm{q}}(\mathcal{C})$ of dimension $n-k$.

A constraint of the form: $G(\mathrm{q}, \dot{\mathrm{q}}, t) < 0$ or $G(\mathrm{q}, \dot{\mathrm{q}}, t) \leq 0$ where G is not integrable, is a nonholonomic inequality constraint. It restricts the set of achievable velocities at any configuration q to a subset of $T_{\mathrm{q}}(\mathcal{C})$ having the same dimension as $T_{\mathrm{q}}(\mathcal{C})$. A

[1] If constraint (1) depends on t, $\mathcal{A}$'s configuration space is time-dependent, otherwise it is time-independent. Many usual holonomic constraints – e.g., the prismatic and revolute joints of a manipulator arm – are time-independent.

constraint bounding the steering angle of a car-like robot is a typical nonholonomic inequality constraint.

A nonholonomic constraint is generally caused by a rolling contact between two rigid bodies. It expresses that the relative velocity of the two points of contact is zero. When the motion in contact combines rolling and sliding, the expression, which depends on the friction coefficient of the two bodies, is nonlinear. When there is no sliding, the nonholonomic constraint is linear in $\dot{\mathbf{q}}$. The second case, although less general than the first, is much simpler and quite widespread in practice. Therefore, in the following, we only consider constraints of the form (2) which are linear in $\dot{\mathbf{q}}$. In the car-like robot example, this corresponds to assuming no slipping of the wheels on the ground.

When dealing with constraints of the form (2), two important questions arise: Are they integrable? If they are not integrable, do they restrict the range of achievable configurations? We investigate these questions in the next two subsections.

2.2 Characterization of Nonholonomy

Any kinematic constraint of the form (2), which is linear in $\dot{\mathbf{q}}$ can we rewritten as follows[2]:

$$G(\mathbf{q},\dot{\mathbf{q}}) = \omega(\mathbf{q})\cdot\dot{\mathbf{q}} = \sum_{i=1}^{i=n}\omega^i(\mathbf{q})\dot{q}_i = 0. \qquad (3)$$

By definition, ω is called a 1-*(differential) form* [Spivak, 1979]. For every $\mathbf{q}\in\mathcal{C}$, equation (3) determines an hyperplane denoted by $\Delta(\mathbf{q})$, which is included in the tangent space $T_{\mathbf{q}}(\mathcal{C})$. $\Delta(\mathbf{q})$ is called the $(n-1)$-*distribution* associated with ω.

Let us now consider the case where there are k constraints of the form (3). Then, $\mathcal{A}$'s motion is constrained by the following system of equations:

$$G_j(\mathbf{q},\dot{\mathbf{q}}) = \omega_j(\mathbf{q})\cdot\dot{\mathbf{q}} = \sum_{i=1}^{i=n}\omega_j^i(\mathbf{q})\dot{q}_i = 0, \quad j=1,\ldots,k. \qquad (4)$$

Let us assume that the equations are independent. In general, at any given time t, for every $\mathbf{q}\in\mathcal{C}$, this system of equations determines a $(n-k)$-dimensional linear subspace $\Delta(\mathbf{q})$ of $T_{\mathbf{q}}(\mathcal{C})$. The subspace $\Delta(\mathbf{q})$ is called the $(n-k)$-*distribution* associated with the k 1-forms $(\omega_1,\ldots,\omega_k)$.

In the presence of k independent constraints as in (4) and under some regularity conditions which we assume to be satisfied (see [Isodori, 1985]), it is always possible to find a set of $n-k$ independent C^∞ vector fields $X_1(\mathbf{q}),\ldots,X_{n-k}(\mathbf{q})$ spanning $\Delta(\mathbf{q})$. Hence, for a robot subject to the constraints in (4), the velocity can always be expressed as a linear combination of $X_1,\ldots,X_{n-k}$.

Let (X,Y) be any pair of vector fields, such that for any $\mathbf{q}\in\mathcal{C}$ both $X(\mathbf{q})$ and $Y(\mathbf{q})$ belong to $\Delta(\mathbf{q})$. Given any configuration $\mathbf{q}$, consider a path of $\mathcal{A}$ starting at $\mathbf{q}$ obtained by concatenating four consecutive paths:
- the first path follows the flow of X during δt;
- the second path follows the flow of Y during δt;
- the third path follows the flow of $-X$ during δt;
- the fourth path follows the flow of $-Y$ during δt.

[2]For simplicity, in the rest of the paper, we assume that the kinematic constraints do not depend on time. However, all the results remain valid when constraints are time-dependent.

We denote by $\mathbf{q}'$ the configuration reached at the end of these four paths. A straightforward Taylor expansion shows that:

$$\lim_{\delta t\to 0}\frac{\mathbf{q}'-\mathbf{q}}{\delta t^2} = dY\cdot X - dX\cdot Y.$$

The expression $dY\cdot X - dX\cdot Y$ determines a new vector field, which is commonly denoted by $[X,Y]$ and called *Lie bracket* of X and Y. Hence, the above motion of $\mathcal{A}$ along vectors of the distribution Δ is biased with $\delta t^2[X,Y]$. A necessary condition for integrability of the distribution Δ is therefore that all the Lie brackets of all vector fields in Δ be in Δ. This condition turns out to be also sufficient, which is precisely the Frobenius Integrability Theorem in its general form [Spivak, 1979]:

THEOREM 1 (Frobenius Integrability Theorem – General Case): *Let Δ be a $(n-k)$-distribution on a n-dimensional manifold $\mathcal{C}$ associated with the k 1-forms $(\omega_1(\mathbf{q}),\ldots,\omega_k(\mathbf{q}))$. In a neighborhood of any point $\mathbf{q}_0\in\mathcal{C}$, the following two conditions are equivalent:*

1. *The distribution Δ is closed under the Lie bracket operation – i.e., for any pair of vector fields (X,Y) in Δ, $[X,Y]$ is also in Δ.*
2. *There is a foliation of $\mathcal{C}$ tangent to Δ – i.e., the constraints $\omega_j(\mathbf{q})\cdot\dot{\mathbf{q}}=0$, $j=1,\ldots,k$ are integrable.*

Unfortunately, this result is stated in terms of vector fields on Δ, not in terms of the $\omega_1,\ldots,\omega_k$. It does not provide an effective way to test the holonomy of the constraints in (4).

When $k=1$, however, the above characterization can be rewritten with ω. Let us consider the case where the motion of $\mathcal{A}$ is constrained by a single constraint of the form (3). Saying that this constraint is integrable is equivalent to saying that there is a function V over $\mathcal{C}$, such that $dV(\mathbf{q}) = \lambda(\mathbf{q})\omega(\mathbf{q})$ for some non-zero integrating factor $\lambda(\mathbf{q})$. Taking the exterior differential of the above equation, we get: $0 = d(dV) = d\lambda\wedge\omega + \lambda d\omega$. Multiplying exteriorly this result by ω, we obtain:

$$\omega\wedge d\omega = 0$$

which is a necessary condition for integrability of equation (3). It turns out that this condition is also sufficient:

THEOREM 2 (Frobenius Integrability Theorem – Case of a Scalar Constraint): *Let $\omega(\mathbf{q})$ be a 1-form on a manifold $\mathcal{C}$ and Δ the associated distribution. In a neighborhood of any point $\mathbf{q}_0\in\mathcal{C}$, the following three conditions are equivalent:*

1. *$\omega\wedge d\omega=0$ – i.e., the exterior product of ω and its exterior differential is null.*
2. *The distribution Δ is closed under the Lie bracket operation– i.e., for any couple of vector fields (X,Y) in Δ, $[X,Y]$ is also in Δ.*
3. *There is a foliation of $\mathcal{C}$ tangent to Δ – i.e. the constraint $\omega(\mathbf{q})\cdot\dot{\mathbf{q}}=0$ is integrable.*

A pedestrian proof of this theorem based on elementary calculus (Fixed point theorem) can be found in [Barraquand, 1988]. As shown in [Spivak, 1979] (pp. 264-268) the local results of Theorems 1 and 2 can be globalized to the whole manifold $\mathcal{C}$ (integral manifold).

By explicitly calculating $\omega\wedge d\omega$, we obtain an effective local characterization of holonomy for a single scalar linear kinematic constraint of the form (3):

COROLLARY 1 (Characterization of linear holonomy for a scalar constraint): *A single scalar linear kinematic constraint defined by:*

$$G(\mathbf{q}, \dot{\mathbf{q}}) = \omega(\mathbf{q}) \cdot \dot{\mathbf{q}} = \sum_{i=1}^{i=n} \omega_i(\mathbf{q})\dot{\mathbf{q}}_i = 0$$

is holonomic iff for any $i, j, k \in [1, n]$ such that $1 \leq i < j < k \leq n$, we have $A_{ijk} = 0$, with:

$$A_{ijk} = \omega_i\left(\frac{\partial \omega_k}{\partial q_j} - \frac{\partial \omega_j}{\partial q_k}\right) + \omega_j\left(\frac{\partial \omega_i}{\partial q_k} - \frac{\partial \omega_k}{\partial q_i}\right) + \omega_k\left(\frac{\partial \omega_j}{\partial q_i} - \frac{\partial \omega_i}{\partial q_j}\right).$$

2.3 Controllability of Nonholonomic Robots

Frobenius Integrability Theorem is the theoretical basis for some major results in controllability theory for nonlinear control systems (e.g., see [Isidori, 1985]). The applicability of these results to the analysis of the controllability of nonholonomic robots was first noticed by Li and Canny [Li and Canny, 1989].

A key concept in controllability theory is the so-called *Control Lie Algebra*. It can be defined as follows. Let Δ be a $(n-k)$-distribution on a n-dimensional integral manifold generated by a set of independent smooth vector fields $X_1, \ldots, X_{n-k}$. The Control Lie Algebra associated with Δ, denoted by $CLA(\Delta)$, is the smallest distribution which contains Δ and is closed under the Lie bracket operation. Stated otherwise, $CLA(\Delta)$ is the distribution generated by $X_1, \ldots, X_{n-k}$ and all their Lie brackets recursively computed. By construction, $CLA(\Delta)$ verifies the conditions of Frobenius Theorem, and is therefore integrable. Obviously, the dimension m of $CLA(\Delta)$ verifies: $m \geq n-k$.

The following theorem derives from the original work of Chow [Chow, 1939], which was subsequently elucidated by several authors (e.g., [Lobry, 1979]):

THEOREM 3 (Controllability of Nonlinear Systems): *Let Δ be a $(n-k)$-distribution on a connected open subset S of a n-dimensional compact manifold C. Let $CLA(\Delta)$ be the Control Lie Algebra associated with Δ. Any two points $\mathbf{q}_1$ and $\mathbf{q}_2$ in S can be connected by a path in S following the distribution Δ if and only if the dimension of $CLA(\Delta)$ is equal to n.*

In other words, this theorem says that the nonlinear system generated by the distribution Δ is fully controllable if and only if its Control Lie Algebra has maximal dimension.

There is an immediate corollary of this theorem which is particularly useful for characterizing the controllability of a robot that is constrained by a single scalar nonholonomic equality relation. The constraint can be represented by a $(n-1)$-distribution Δ generated by $\{X_1, \ldots, X_{n-1}\}$. According to the Frobenius Theorem, for each $\mathbf{q} \in C$, there must exist at least one pair of integers $i, j \in [1, n-1]$ such that the Lie bracket $[X_i, X_j]$ does not belong to Δ. Therefore, the Control Lie Algebra has a dimension m strictly greater than $n-1$. Since it cannot be greater than n, it is equal to n. Hence:

COROLLARY 2 (Controllability with a Single Scalar Linear Nonholonomic Equality Constraint): *Any robot, which is subject to a single scalar linear nonholonomic equality constraint is fully controllable - i.e., any two points lying in an open connected susbset of the configuration space can be connected by a path lying in this subset and respecting the kinematic constraint.*

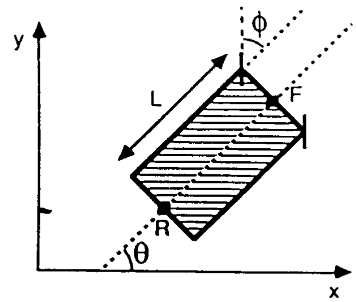

Figure 1: Car-Like Robot

3 Application to Two Examples

3.1 Car-Like Robot

Consider a front-wheel-drive four-wheel car[3]. We model the car by a two-dimensional object translating and rotating in the plane, as illustrated in Figure 1. The configuration space of the car is $D \times S^1$, where D is a compact domain of $\mathbb{R}^2$. We parameterize the car configuration by the coordinates X_f and Y_f of the midpoint F between the two front wheels and the angle θ between the x axis of the Cartesian frame embedded in the plane and the main axis of the car. The velocity parameters are $\dot{X}_f$, $\dot{Y}_f$ and $\dot{\theta}$. The control parameters of the car are the velocity $v \in \mathbb{R}$ of the front wheels (if $v > 0$, the car moves forward) and the steering angle ϕ measuring the orientation of the front wheels with respect to the main axis of the car.

Let R be the midpoint between the two rear wheels (see Figure 1) and (X_r, Y_r) its coordinates. Assuming a pure rolling contact between the wheels and the ground – i.e., no slipping – the velocity of R is always parallel to the main axis of the car. Hence, we have:

$$\dot{X}_r = \lambda \cos\theta \qquad \dot{Y}_r = \lambda \sin\theta.$$

Eliminating λ, we get:

$$-\dot{X}_r \sin\theta + \dot{Y}_r \cos\theta = 0. \qquad (5)$$

The coordinates (X_f, Y_f) are related to (X_r, Y_r) by:

$$X_r = X_f - L\cos\theta \qquad Y_r = Y_f - L\sin\theta$$

where L denotes the distance between R and F. Thus:

$$\dot{X}_r = \dot{X}_f + \dot{\theta}L\sin\theta \qquad \dot{Y}_r = \dot{Y}_f - \dot{\theta}L\cos\theta$$

Combining these relations with (5), we obtain the following kinematic constraint:

$$-\dot{X}_f \sin\theta + \dot{Y}_f \cos\theta - \dot{\theta}L = 0 \qquad (6)$$

Using the notations of Subsection 2.2, we identify:

$$\omega_1 = -\sin\theta \quad \omega_2 = \cos\theta \quad \omega_3 = -L$$

Applying Corollary 1, we compute the (unique) coefficient A_{123} and we get $A_{123} = -1 \neq 0$. Therefore, *the kinematic constraint (6) is nonholonomic.*

[3]Our presentation can easily be modified to treat other types of car-like robots.

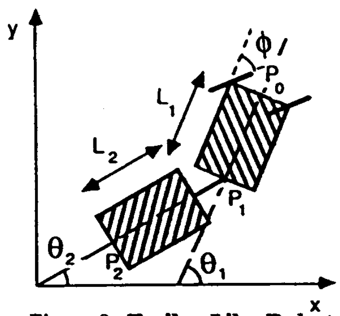

Figure 2: Trailer-Like Robot

Assuming no bound on the steering angle ϕ, this kinematic constraint is the only one applying to the motions of the car. Therefore, according to Corollary 2 (Subsection 2.3), *the car is fully controllable.*

The velocity parameters are:

$$\left.\begin{array}{rcl} \dot{X}_f & = & v\cos(\theta+\phi) \\ \dot{Y}_f & = & v\sin(\theta+\phi) \\ \dot{\theta} & = & v\frac{\sin\phi}{L} \end{array}\right\} \qquad (7)$$

where the third relation is derived from the first two and constraint (6).

At any configuration (X_f, Y_f, θ), the velocity parameters span a two-dimensional subspace. The velocity is completely determined by the configuration and, say, $\dot{X}_f$ and $\dot{Y}_f$. If the steering angle ϕ is not bounded, all the velocity directions are achievable in this two-dimensional space. If ϕ is bounded – e.g., $|\phi| \leq \phi_{max}$, with $\phi_{max} < \frac{\pi}{2}$ – then only a subset of the velocity orientations is feasible.

The constraint bounding the values of ϕ translates into a nonholonomic nonlinear inequality constraint on the motions of the car. Corollary 2 does not apply any longer. Nevertheless, it has been shown by Laumond [Laumond, 1986] that the car remains fully controllable. Although the proof given by Laumond is specific to car-like robots, this result suggests that it might be possible to give a more general characterization of the controllability of nonholonomic robots than Corollary 2, which would be applicable when there are nonholonomic inequality constraints in addition to the nonholonomic equation.

3.2 Trailer-Like Robot

The trailer problem is an extension of the car problem, obtained by adding one or more bodies to be towed by the car. A n-body trailer system consists of a car towing $n-1$ bodies serially hooked (e.g., a luggage carrier in an airport). Figure 2 displays a schematic model of a two-body trailer. The midpoint between the front wheels of the first body (the car) is denoted by P_0. The midpoint between the rear wheels of the k^{th} body is denoted by P_k. We therefore have $(n+1)$ points $P_0, \ldots, P_n$, whose coordinates are denoted by $(X_0, Y_0), \ldots, (X_n, Y_n)$. The orientation of the k^{th} body with respect to the x axis of the Cartesian frame embedded in the plane is denoted by θ_k. The configuration space of the n-body trailer is $D \times (S^1)^n$, where D is a compact domain of $\mathbb{R}^2$. We parameterize the trailer configuration by $(X_0, Y_0, \theta_1, \ldots, \theta_n)$. The velocity parameters are $\dot{X}_0, \dot{Y}_0, \dot{\theta}_1, \ldots, \dot{\theta}_n$. The control parameters are the same as for the car-like robot, that is, the velocity v and the steering angle ϕ.

There are n kinematic constraints, one for each body. In order to establish these constraints, it is convenient to represent the points $P_0, \ldots, P_n$ in the complex plane, i.e.: $P_k = X_k + iY_k$. Denoting by L_k the length of the k^{th} body, we can write the geometric constraint between the bodies $k-1$ and k as:

$$P_k = P_{k-1} - L_k \exp(i\theta_k) = P_0 - \sum_{l=1}^{k} L_l \exp(i\theta_l) \qquad (8)$$

The kinematic constraint of the k^{th} body is $\dot{P}_k = \lambda \exp(i\theta_k)$, which is equivalent to $\Im(\exp(-i\theta_k)\dot{P}_k) = 0$, where $\Im(z)$ denotes the imaginary part of the complex number z. Combining this characterization with the derivative of equation (8) and using the linearity of the $\Im$ operator, we obtain the k^{th} kinematic constraint:

$$L_k\dot{\theta}_k = -\dot{X}_0 \sin\theta_k + \dot{Y}_0 \cos\theta_k - \sum_{l=1}^{k-1} L_l \dot{\theta}_l \cos(\theta_l - \theta_k).$$

In particular, for $k = 1$, we get:

$$L_1\dot{\theta}_1 = -\dot{X}_0 \sin\theta_1 + \dot{Y}_0 \cos\theta_1 \qquad (9)$$

which is precisely the constraint (6) of the car problem.

For $k = 2$, we get:

$$L_2\dot{\theta}_2 = -\dot{X}_0 \sin\theta_1 + \dot{Y}_0 \cos\theta_1 - L_1\dot{\theta}_1 \cos(\theta_2 - \theta_1) \qquad (10)$$

Equations (9) and (10) are the kinematic constraints of the two-body trailer system.

Thus, the equations governing the motion of the two-body trailer system are:

$$\left.\begin{array}{rcl} \dot{X}_0 & = & v\cos(\theta_1+\phi) \\ \dot{Y}_0 & = & v\sin(\theta_1+\phi) \\ \dot{\theta}_1 & = & v\frac{\sin\phi}{L_1} \\ \dot{\theta}_2 & = & v\frac{\cos\phi\sin(\theta_1-\theta_2)}{L_2} \end{array}\right\} \qquad (11)$$

where the third and fourth relations are derived from the first two and constraints (9) and (10).

We now analyze the integrability of the kinematic constraints applying to the motions of the two-body trailer system. Since there are two independent kinematic constraints, we cannot use the simple charaterization of holonomy given by Corollary 1. However, the Frobenius Theorem (general case) still applies. As suggested in [Li and Canny, 1989], we can compute the dimension m of the Control Lie Algebra associated with the two constraints. We show below that m is maximal, i.e. $m = n = 4$. Applying Theorem 1, this result directly entails that *the two-body trailer system is non-holonomic.* Applying Theorem 3, it entails that *the two-body trailer system is fully controllable.*

PROPOSITION 1: *The Control Lie Algebra associated with the two kinematic constraints of the two-body trailer system has maximal dimension $m = 4$.*

Proof: A straightforward computation shows that the following two vector fields satisfy both constraints (9) and (10):

$$\begin{array}{rcl} X_1 & = & (-L_1\sin\theta_1 \quad L_1\cos\theta_1 \quad 1 \quad 0)^T \\ X_2 & = & (L_1\cos\theta_1 \quad L_1\sin\theta_1 \quad 0 \quad (L_1/L_2)\sin(\theta_1-\theta_2))^T \end{array}$$

Let $X_3 = [X_1, X_2]$ and $X_4 = [X_2, X_3]$. We compute:

$$\begin{array}{rcl} X_3 & = & (-L_1\sin\theta_1 \quad L_1\cos\theta_1 \quad 0 \quad (L_1/L_2)\cos(\theta_1-\theta_2))^T \\ X_4 & = & (0 \quad 0 \quad 0 \quad (L_1/L_2)^2)^T \end{array}$$

We have $det(X_1, X_2, X_3, X_4) = \frac{L_1^4}{L_2^2} > 0$. Hence, X_1, X_2, X_3 and X_4 are four independent vector fields and the Control Lie Algebra has dimension $m = 4$. ■

4 Planning with Nonholonomic Constraints

4.1 Overview of the Planner

Let the workspace $\mathcal{W}$ of a robot $\mathcal{A}$ be populated by some stationary obstacles $\mathcal{B}_i$, $i = 1, ..., q$. These obstacles map in the configuration space $\mathcal{C}$ of $\mathcal{A}$ to regions $\mathcal{CB}_i$ called C-obstacles and defined by $\mathcal{CB}_i = \{q \in \mathcal{C} \mid \mathcal{A}(q) \cap \mathcal{B}_i \neq \emptyset\}$, where $\mathcal{A}(q)$ denotes the region of $\mathcal{W}$ occupied by $\mathcal{A}$ at configuration q. The subset $\mathcal{C}_{free} = \mathcal{C} \setminus \bigcup_{i=1}^{n} \mathcal{CB}_i$ is called free space. If both $\mathcal{A}$ and the $\mathcal{B}_i$'s are modelled as closed regions, the $\mathcal{CB}_i$'s are closed; $\mathcal{C}_{free}$ is an open subset of $\mathcal{C}$, hence a manifold of dimension n.

Given two configurations q_1 and q_2 in $\mathcal{C}_{free}$, the path planning problem is to construct a path connecting q_1 to q_2 and lying in $\mathcal{C}_{free}$, i.e. a map $\tau : s \in [0,1] \mapsto \tau(s) \in \mathcal{C}_{free}$, such that $\tau(0) = q_1$ and $\tau(1) = q_2$. In the presence of nonholonomic constraints, the tangent to this path, $\frac{d\tau}{ds}$, must lie in the subset of the tangent space of $\mathcal{C}$ selected out by the constraints.

We have implemented a general-purpose path planner based on the following main ideas [Barraquand and Latombe, 1989]:

- $\mathcal{C}$ is discretized and explored in a trial-and-error fashion. The exploration is guided by potential functions using a classical best-first search algorithm.
- $\mathcal{W}$ is used as a major source of inspiration for building potential functions with "good" characteristics, i.e. with few local minima or local minima having small domains of attraction.
- $\mathcal{W}$ is represented as a bitmap (distributed representation), which allows us to apply very efficient algorithms for computing the potential functions and checking collisions.

We report the reader to [Barraquand and Latombe, 1989] for a detailed presentation of the planner.

We have experimented with this planner using a variety of simulated robots, including holonomic mobile robots and manipulator arms with many (8 and 10) degrees of freedom. We have also run several experiments using simulated car-like and trailer-like robots. For these robots, interesting experiments were carried out with a version of the planner specifically designed to minimize the number of maneuvers.

The planner operates in a best-first fashion over a discretized representation of the configuration space. The "normal" version of the planner makes use of a potential function defined over the configuration space. Basically, it follows the negated gradient of this function until it reaches the goal configuration or a local minimum of the potential. It escapes local minima, either by filling them up to the lowest saddle points (when there is a small number of degrees of freedom), or by generating Brownian motions (when there is a large number of degrees of freedom). Using these simple techniques, the planner was able to plan the motions of holonomic mobile robots among complex obstacles in 1 to 3 seconds[4] and the motions of manipulator arms with 8 and 10 degrees of freedom in 1 to 5 minutes.

[4] All the experiments reported in this paper were conducted on a MIPS-based DEC 3100 Workstation.

The version of the planner aimed at minimizing the number of maneuvers does not use any potential function. It applies the same best-first search strategy starting at q_1, but the cost function is the number of maneuvers. The algorithm maintains two lists of configurations, CLOSED and OPEN. CLOSED contains all the configurations whose successors in the discretized configuration space have already been generated. OPEN contains all the attained configurations whose successors have not been generated yet. CLOSED is simply represented by marking the corresponding cells of a large n-dimensional array (of the order of 128^3 for a car-like robot and slightly less than 64^4 for a trailer-like robot). Hence, the access time to CLOSED is constant. OPEN (which is much smaller) is represented as a heap. Every modification and access to OPEN is made in logarithmic time.

In order to generate the successors of a configuration, the planner discretizes the control parameters and, for each set of values of these parameters, it integrates the velocity parameters of the robot along a short distance. For example, for a car-like robot, the integration of the velocity parameters $\dot{X}_f$, $\dot{Y}_f$ and $\dot{\theta}$ (see relations (7)) yields:

$$\theta(t) = \theta(0) + t\frac{v \sin\phi}{L}$$

$$X_f(t) = X_f(0) + \frac{L}{\sin\phi}\left(\sin(\phi + \theta(0) + t\frac{v\sin\phi}{L}) - \sin(\phi + \theta(0))\right)$$

$$Y_f(t) = Y_f(0) - \frac{L}{\sin\phi}\left(\cos(\phi + \theta(0) + t\frac{v\sin\phi}{L}) - \cos(\phi + \theta(0))\right)$$

The planner generates six successors of a configuration by successively setting the values of the two control parameters v and ϕ to the six values in $\{-v_0, v_0\} \times \{-\phi_{max}, 0, +\phi_{max}\}$. The integration time is 1 and v_0 is set to approximately twice the discretization interval of the X_f and Y_f parameters.

In the case of the trailer-like robot, the generation of the successors of a configuration is slightly more involved. While the first three equations in (11) can be integrated analytically when the values of the control parameters (v and ϕ) are constant, this is not the case for the last equation. The planner solves this equation using a fourth order Runge-Kutta method. The paths thus generated do not exactly satisfy the second nonholonomic constraint.

The array in which CLOSED is represented is in fact an array of parallelepipedic neighborhoods in the paramaterized configuration space. Whenever the planner generates a new configuration, it determines the parallelepipedic neighborhood to which the configuration belongs. If the neighborhood is marked (hence, is in CLOSED), the configuration is discarded. Otherwise, the configuration (not the neighborhood) is recorded in OPEN. Hence, the planner never explores from the same neighborhood twice, but it exactly records the discretized path traced by the search. Collisions are checked by intersecting the robot at every attained configuration with the obstacles in the workspace. As the workspace is represented in bitmap form, the test of intersection is very quick and independent of the number of obstacles [Barraquand and Latombe, 1989].

In theory, if a path exists between two input configurations, the planner ultimately generates a path with minimal number of maneuvers (at the resolution of the configuration space discretization). Despite conceptual simplicity of the brute force algorithm applied, the planner was able to solve all the prob-

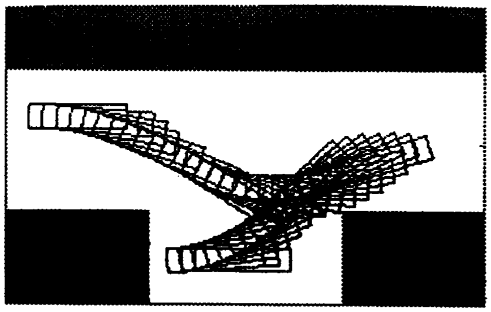

Figure 3: **A Parallel Parking Maneuver**

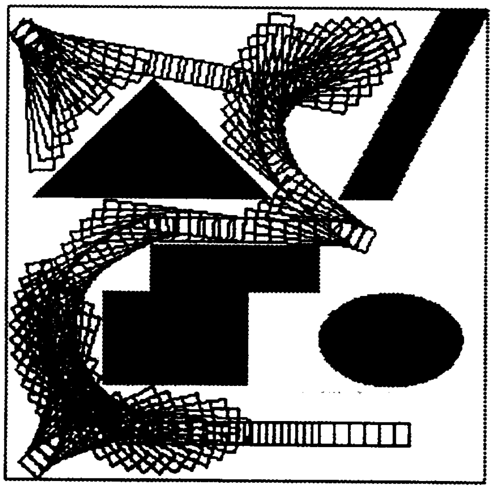

Figure 4: **Maneuvering in Cluttered Workspace**

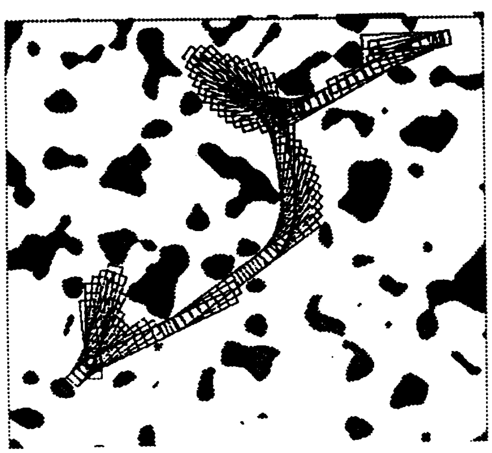

Figure 5: **Maneuvering in Unstructured Workspace**

lems submitted to it, with the configuration space discretization mentionned above. The problems were solved in reasonable amount of time, typically a few minutes (see the next two subsections for more detail). Most of them are non-trivial and would require significant effort for a human to solve. It is clear, however, that this version of the planner is only applicable to robots whose configuration spaces have small dimension. The two-body trailer-like robot, which has a four-dimensional configuration space, stands very close to the practical limit of the planner.

The normal version planner, which uses potential functions to guide the search, is more time and space efficient, but it no longer minimizes the number of maneuvers. In practical applications, we would probably have to compromise between the time devoted to planning and the number of maneuvers[5]. However, we have no simple systematic solution for making such a compromise.

4.2 Results with the Car-Like Robot

We experimented with the planner using a simulated car-like robot with various values of the maximal steering angle ϕ_{max} and several workspaces.

Figure 3 shows an example of the parallel parking problem with a very limited steering angle ($\phi_{max} = 30$ degrees). The running time for that example was 30 seconds.

Figure 4 shows an example with backing-up maneuvers in a cluttered workspace when the maximal steering angle ϕ_{max} is 45 degrees. The running time was about 1 minute. Ten maneuvers (i.e. changes of the sign of v) were necessary in this example.

Figure 5 shows an example of maneuvering in an unstructured workspace represented as a 512^2 bitmap with the same maximal steering angle ϕ_{max} (45 degrees). The running time was about 2 minutes. Four maneuvers were necessary in this example.

4.3 Results with the Trailer-Like Robot

We also conducted several experiments with a simulated two-body trailer-like robot.

Figure 6 shows an example where the trailer has to be parked with a very limited steering angle ($\phi_{max} = 30$ degrees). The running time was 2 minutes.

Figure 7 shows an example where the trailer has to maneuver in a cluttered workspace with a maximal steering angle ϕ_{max} equal to 45 degrees. The running time was about 5 minutes.

5 Conclusion

In this paper, we have presented an implemented path planner, which is able to generate complex paths of nonholonomic mobile robots among obstacles. The generated paths have minimal number of backing-up maneuvers. The approach taken in the planner essentially consists of discretizing both the workspace and the configuration space of the robot, and performing a dynamic programming search in the discretized configuration space. The bitmap representation of the workspace allows

[5] Furthermore, it is easy to create examples for which minimizing the number of maneuvers leads to absurdly long paths.

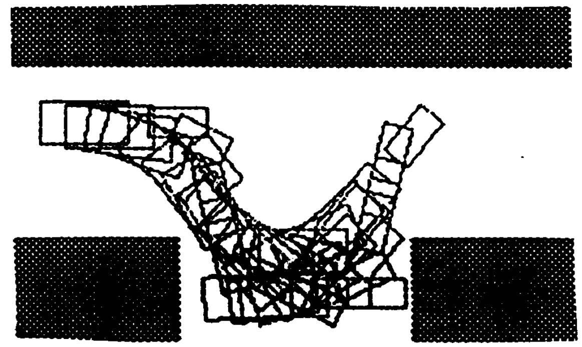

Figure 6: Parking a Trailer

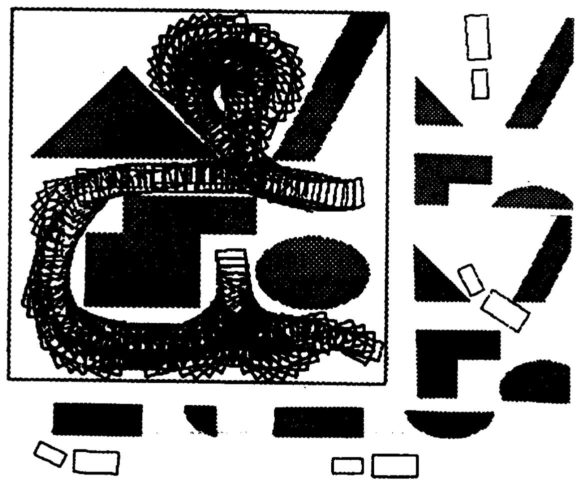

Figure 7: Trailer Maneuvering

the planner to consider any distribution of obstacles in the workspace, with no limitation on the shape or the number of obstacles.

Prior to the presentation of the planner, we proved the controllability of the car-like and trailer-like robots, using general results from differential geometry and nonlinear control theory. These results can also be applied to other nonholonomic robots.

The implemented planner has solved several non-trivial planning problems for car-like robots and trailer-like robots, with limited steering wheel angle. Since it operates in a very systematic fashion, the planner can solve any problem with a reasonable discretization of the configuration space. However, major improvements of the approach are necessary to deal with significantly finer discretizations and higher-dimensional configuration space. Allowing a non-optimal, but still reasonable, number of maneuvers and guiding the search for a path by appropriate potential functions as in [Barraquand and Latombe, 1989] is certainly a promising direction, although it is still not clear how it can be done in a systematic way.

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