Linear Algebra
Prerequisites - continued

Jana Kosecka
http://cs.gmu.edu/~kosecka/cs682.html
kosecka@cs.gmu.edu

Matrices

meaning $A \in \Re^{n \times m}$ n x m matrix

m points from n-dimensional space

$C = AA^T$

Covariance matrix – symmetric
Square matrix associated with
The data points (after mean
has been subtracted)

$$C = \begin{bmatrix} \sum_1^n x_i^2 & \sum_1^n x_i y_i \\ \sum_1^n x_i y_i & \sum_1^n y_i^2 \end{bmatrix}$$

transformation

$A \in \Re^{2 \times 2}$

$\mathbf{y} = A\mathbf{x}$

Special case
matrix is square

Linear equations

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

When is RHS a linear combination of LHS

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Solving linear equations $A\mathbf{x} = \mathbf{y}$

If matrix is invertible $A^{-1}A\mathbf{x} = A^{-1}\mathbf{y}$

$det(A) \neq 0$ $\mathbf{x} = A^{-1}\mathbf{y}$

Linear Equations

Vector space spanned by columns of A $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four basic subspaces

- Column space of A – dimension of C(A)
  number of linearly independent columns
  r = rank(A)
- Row space of A - dimension of R(A)
- number of linearly independent rows
- r = rank(A^T)
- Null space of A - dimension of N(A) n - r
- Left null space of A – dimension of N(A^T) m - r

Linear Equations – Square Matrices

1. A is square and invertible
2. A is square and non-invertible

1. System Ax = b has at most one solution – columns are linearly independent rank = n
   - then the matrix is invertible
2. Columns are linearly dependent rank < n
   - then the matrix is not invertible

Linear Equations – non-square matrices

Long-thin matrix over-constrained system

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad a\mathbf{x} = b$$

The solution exist when b is aligned with [2,3,4]^T

If not we have to seek some approximation – least squares
Approximation – minimize squared error

$$e^2 = (2x - b_1)^2 + (3x - b_2)^2 + (4x - b_3)^2$$

Least squares solution - find such value of x that the error
Is minimized (take a derivative, set it to zero and solve for x)
Short for such solution

$$a\mathbf{x} = b$$
$$a^T a\mathbf{x} = a^T b$$
$$\bar{\mathbf{x}} = \frac{a^T b}{a^T a}$$

Linear equations – non-squared matrices

Similarly when A is a matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$A\mathbf{x} = b$$
$$A^T A\mathbf{x} = A^T b$$
$$\bar{\mathbf{x}} = (A^T A)^{-1} A^T b$$

- If A has linearly independent columns $A^T A$ is square, symmetric and invertible

Eigenvalues and Eigenvectors

- Motivated by solution to differential equations
- For square matrices $A \in \Re^{n \times n}$ $\quad \dot{\mathbf{u}} = A\mathbf{u}$ $\qquad A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

For scalar ODE's

$$\dot{u} = au$$
$$u(t) = e^{at}u(0)$$

We look for the solutions of the following type exponentials

$$v(t) = e^{\lambda t} y$$
$$w(t) = e^{\lambda t} z$$

Substitute back to the equation

$$\lambda e^{\lambda t} y = 4e^{\lambda t} y - 5e^{\lambda t} z$$
$$\lambda e^{\lambda t} z = 2e^{\lambda t} y - 3e^{\lambda t} z$$

$$\mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix} \qquad \lambda\mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x}$$

Eigenvalues and Eigenvectors

$$\lambda \mathrm{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathrm{x} \qquad A\mathrm{x} = \lambda \mathrm{x}$$

eigenvalue (λ), eigenvector (x)

Solve the equation: $(A - \lambda I)\mathrm{x} = 0$ (1)

x – is in the null space of $(A - \lambda I)$
λ is chosen such that $(A - \lambda I)$ has a null space

Computation of eigenvalues and eigenvectors (for dim 2,3)

1. Compute determinant
2. Find roots (eigenvalues) of the polynomial such that determinant = 0
3. For each eigenvalue solve the equation (1)

For larger matrices – alternative ways of computation

Eigenvalues and Eigenvectors

For the previous example

$$\lambda_1 = -1, x_1 = [1, 1]^T \qquad \lambda_2 = -2, x_2 = [5, 2]^T$$

We will get special solutions to ODE $\dot{\mathrm{u}} = A\mathrm{u}$

$$A\mathrm{u} = e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathrm{u} = e^{\lambda_2 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Their linear combination is also a solution (due to the linearity of $\dot{\mathrm{u}} = A\mathrm{u}$)

$$\mathrm{u} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_1 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

In the context of diff. equations – special meaning
Any solution can be expressed as linear combination
Individual solutions correspond to modes

Eigenvalues and Eigenvectors

$$A\mathrm{x} = \lambda \mathrm{x}$$

Only special vectors are eigenvectors
- such vectors whose direction will not be changed by the transformation A (only scale)
- they correspond to normal modes of the system act independently

Examples	eigenvalues	eigenvectors
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$	2, 3	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Whatever A does to an arbitrary vector is fully determined by its eigenvalues and eigenvectors

$$A\mathrm{x} = 2\lambda_1 v_1 + 5\lambda_2 v_2$$

Eigenvalues and Eigenvectors - Diagonalization

- *Given a square matrix A and its eigenvalues and eigenvectors – matrix can be diagonalized*

$$A = S \Lambda S^{-1}$$

$$A = S \Lambda S^{-1}$$

Matrix of eigenvectors (S), Diagonal matrix of eigenvalues (Λ)

$$AS = \Lambda S$$

$$A \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \ldots & \lambda_n x_n \end{bmatrix} \qquad A\mathrm{x} = \lambda \mathrm{x}$$

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \ldots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \ldots \\ & & \lambda_n \end{bmatrix}$$

$$A = S \Lambda S^{-1}$$

- If some of the eigenvalues are the same eigenvectors are not independent

Diagonalization

- If there are no zero eigenvalues – matrix is invertible
- If there are no repeated eigenvalues – matrix is diagonalizable
- If all the eigenvalues are different then eigenvectors are linearly independent

For Symmetric Matrices

If A is symmetric $A = Q\Lambda Q^T$

Diagonal matrix of eigenvalues

orthonormal matrix of eigenvectors

i.e. for a covariance matrix
or some matrix B = A^TA