

## Linear Algebra Basics

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## Why do we need Linear Algebra?

- We will associate coordinates to
  - 3D points in the scene
  - 2D points in the CCD array
  - 2D points in the image
- Coordinates will be used to
  - Perform geometrical transformations
  - Associate 3D with 2D points
- Images are matrices of numbers
  - We will find properties of these numbers

Most of the slides in this review are courtesy O. Camps (Penn State Univ.)

## Matrices

$$A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n \times m} = A_{n \times m} + B_{n \times m}$$
$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

## Matrices

Product:

$$C_{n \times p} = A_{n \times m} B_{m \times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$$

$$A_{n \times n} B_{n \times n} \neq B_{n \times n} A_{n \times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

## Matrices

Transpose:

$$C_{m \times n} = A^T_{n \times m}$$

$$(A + B)^T = A^T + B^T$$

$$c_{ij} = a_{ji}$$

$$(AB)^T = B^T A^T$$

If  $A^T = A$  **A is symmetric**

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \quad \begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

## Matrices

Determinant: **A must be square**

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example:  $\det \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

## Matrices

Inverse:

**A must be square**

$$A_{n \times n} A^{-1}_{n \times n} = A^{-1}_{n \times n} A_{n \times n} = I$$

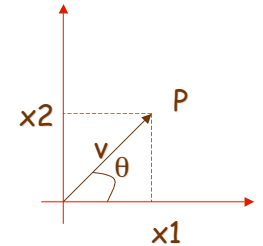
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example:  $\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

## 2D,3D Vectors

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \quad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$



Magnitude:  $\|\mathbf{v}\| = \sqrt{x_1^2 + x_2^2}$

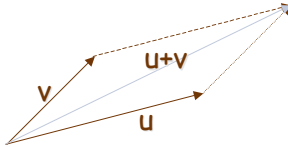
If  $\|\mathbf{v}\| = 1$ , **v** is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left( \frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right) \text{ Is a unit vector}$$

Orientation:  $\theta = \tan^{-1} \left( \frac{x_2}{x_1} \right)$

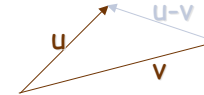
## Vector Addition

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



## Vector Subtraction

$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$



## Scalar Product

$$av = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \end{bmatrix}$$



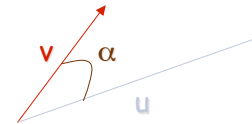
## Inner (dot) Product

$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 \cdot v_1 + u_2 \cdot v_2$$

The inner product is a **SCALAR!**

$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \|u\| \|v\| \cos \alpha$$

$$u^T v = 0 \leftrightarrow u \perp v$$

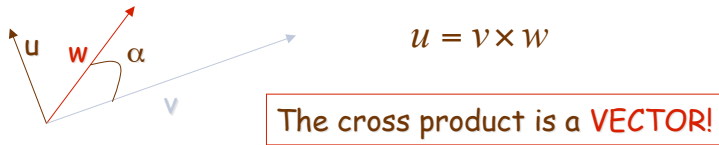


$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\langle u, v \rangle \doteq u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3 \quad \cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

$$\|u\| \doteq \sqrt{u^T u} = \sqrt{u_1^2 + u_2^2 + u_3^2} \quad \text{norm of a vector}$$

## Vector (cross) Product



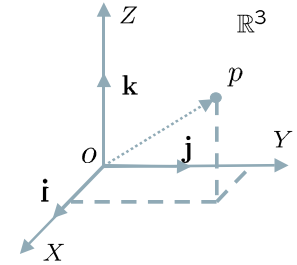
Magnitude:  $\|u\| = \|v \cdot w\| = \|v\| \|w\| \sin \alpha$

Orientation:  $u \perp v \rightarrow u^T v = (u \times v)^T v = 0$   
 $u \times v = -v \times u$   
 $a(u \times v) = au \times v = u \times av$   
 $u \parallel u \rightarrow (u \times v) = 0$

## Orthonormal Basis in 3D

Standard base vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



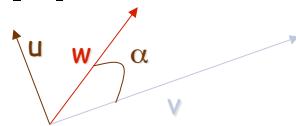
Coordinates of a point  $p$  in space:

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3 \quad \mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = X \cdot \mathbf{i} + Y \cdot \mathbf{j} + Z \cdot \mathbf{k}$$

## Vector (Cross) Product Computation

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

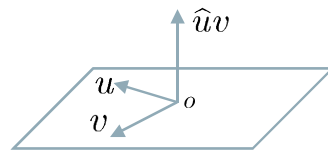
$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$



$$= (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

$$u \times v \doteq \hat{u}v, \quad u, v \in \mathbb{R}^3$$

$$\hat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

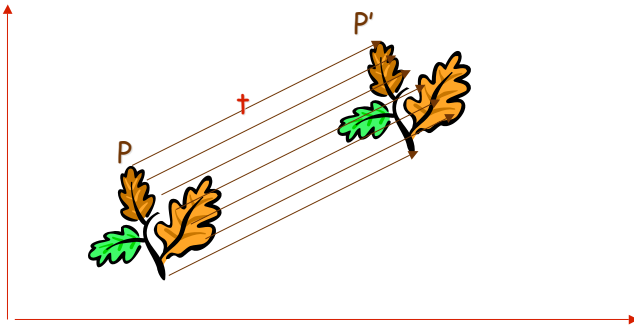


Skew symmetric matrix associated with vector

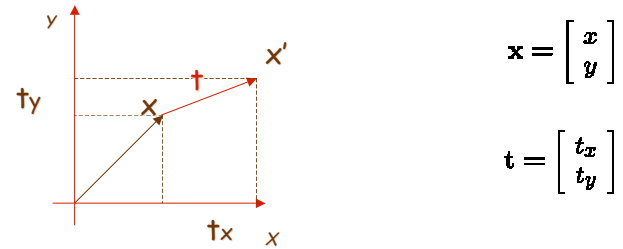
$$\hat{u} = -(\hat{u})^T$$

## 2D Geometrical Transformations

## 2D Translation



## 2D Translation Equation



$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t} = \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix}$$

## Homogeneous Coordinates

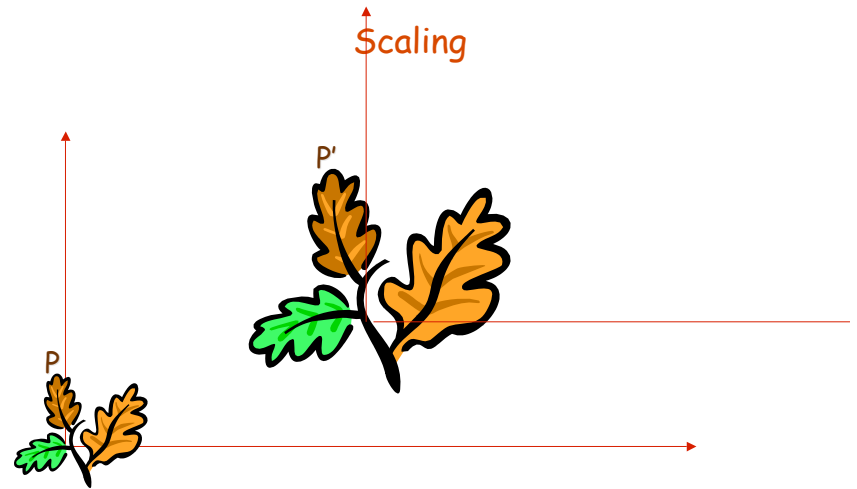
Homogeneous coordinates:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathbb{R}^3,$$

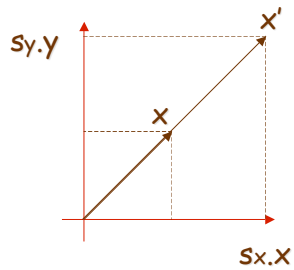
Translation using matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = P_t \mathbf{x}$$



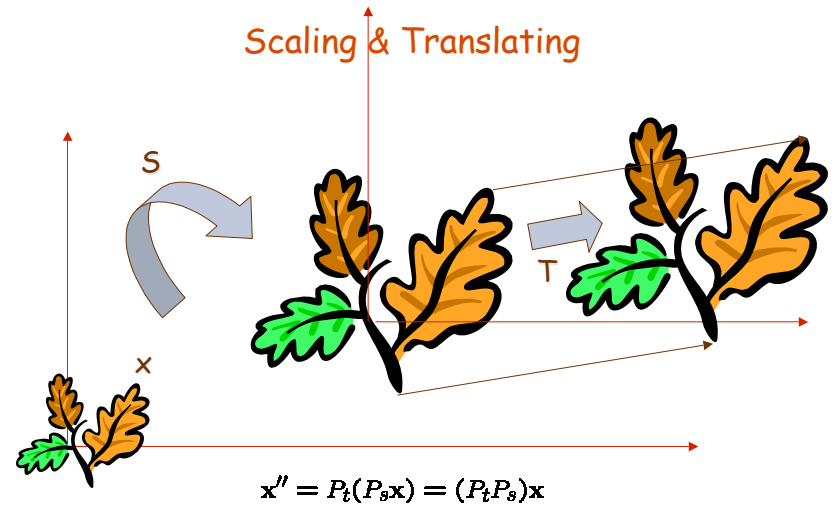
## Scaling Transformation



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = P_s \mathbf{x}$$

## Scaling & Translating



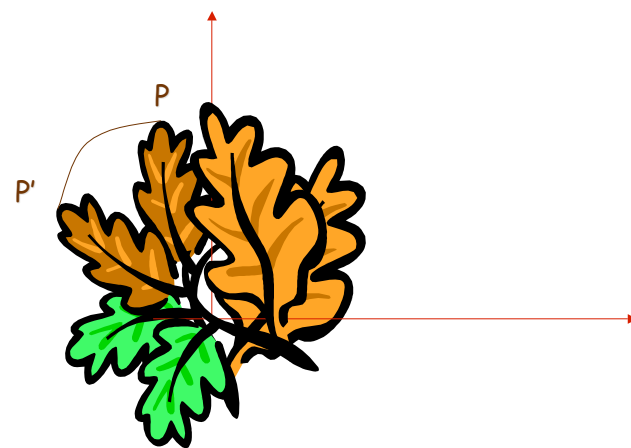
Composition of transformations

## Scaling & Translating

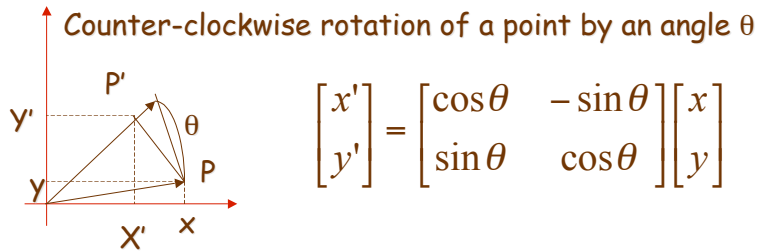
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

## Rotation in 2D

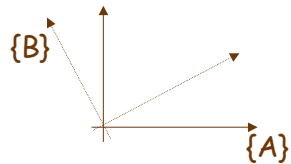


## Rotation Matrix



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Counter-clockwise rotation of a coordinate frame attached to a rigid body by an angle  $\theta$



Interpretations of the rotation matrix  $R_{BA}$

Columns of  $R_{BA}$  are the unit vectors of the axes of frame B expressed in coordinate frame A. Such rotation matrix transforms coordinates of points in frame A to points in frame B

## Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2  $\rightarrow$  4 elements

BUT! There is only 1 degree of freedom:  $\theta$

The 4 elements must satisfy the following constraints:

$$\begin{aligned} R \cdot R^T &= I \text{ Rows and columns are orthogonal and of unit length} \\ \det(R) &= 1 \text{ Matrix is orientation preserving} \end{aligned}$$

## Scaling, Translating & Rotating

Rotation matrix in 2D in homogeneous coordinates

$$R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Order of multiplication matters!

$$X' = P_s \cdot X$$

$$X'' = P_t \cdot X' = (P_t \cdot P_s) \cdot X$$

$$X''' = R \cdot X'' = R \cdot (P_t \cdot P_s) \cdot X$$

$$R \cdot P_t \cdot P_s \neq R \cdot P_s \cdot P_t$$

Next transformations in 3D